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A priori error estimates for elliptic optimal control problems with a bilinear state equation

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ABSTRACT

In this paper a priori error analysis for the finite element discretization of an optimal control problem governed by an elliptic state equation is considered. The control variable enters the state equation as a coefficient and is subject to pointwise inequality constraints. We derive a priori error estimates for the discretization error in the control variable and confirm our theoretical results by numerical examples.

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1. Introduction

In this paper we present a priori error analysis for the finite element discretization of an optimal control problem with an elliptic equation. The control variable enters the state equation as a coefficient. We consider the following optimal control problem for the state u and the control q involving pointwise control constraints:

$$\begin{cases} \text{Minimize } J(q,u) = \dfrac{1}{2}\displaystyle\int_\Omega (u-u_d)^2 \mathrm{d}x + \dfrac{\alpha}{2}\displaystyle\int_\Omega (q-q_d)^2 \mathrm{d}x, \text{ s.t.} \\ -\Delta u + qu = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ \text{with } 0 < a \le q \le b \quad \text{a.e. in } \Omega. \end{cases} \tag{P}$$

A precise formulation including a function analytical setting is given in the next section. Since q enters the state equation as a coefficient, we can also interpret the problem as a parameter estimation problem.

On one hand there are only few publications dealing with error estimates for distributed parameter identification problems governed by elliptic partial differential equations, see [1–8]. However, the problems considered there are quite different to the optimal control problem under consideration.

On the other hand there are many publications dealing with a priori estimates for optimal control problems, see for elliptic problems, e.g., [9–15] and for parabolic problems, e.g., [16–18]. In these publications the error $\|\bar{q}-\bar{q}_h\|_{L^2(\Omega)}$ between the solution $\bar{q}$ of a continuous problem and the solution $\bar{q}_h$ of the discretized one is analyzed. However, in all these publications the control variable enters the state equation on the right-hand side or is part of the boundary condition. In [9,13] the convergence order $\|\bar{q}-\bar{q}_h\|_{L^2(\Omega)} = \mathcal{O}(h)$ was shown using a cellwise constant discretization of the control variable. For the finite element discretization of the control by (bi-/tri-)linear H^1-conforming elements, the convergence order $\mathcal{O}(h^{\frac{3}{2}})$ was verified, see, e.g., [10]. There are two approaches to prove $\mathcal{O}(h^2)$-convergence for the error in the control variable in

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the presence of control constraints, see [14,15]. In [14] a variational approach is proposed without explicitly discretizing the control variable and in [15] a post-processing step is used to obtain the desired order of convergence. In [17,18] similar estimates were established for parabolic equations.

The main purpose of this paper is to analyze the error $\|\bar{q}-\bar{q}_h\|_{L^2(\Omega)}$ with respect to the discretization parameter h. The variable $\bar{q}$ stands for a fixed locally optimal control of (P) and $\bar{q}_h$ is an associate one of (P_h) being an approximate optimal control problem which we obtain by a standard finite element discretization. The solution of the state equation depends nonlinearly on the control variable. Therefore, we cannot guarantee uniqueness of the solution of the optimization problem and hence, we concentrate on locally optimal controls. They are the natural results of numerical optimization algorithms. For a given locally optimal control $\bar{q}$ of (P) we prove that there exists a sequence $(\bar{q}_h)_{h>0}$ of locally optimal controls of (P_h) converging to $\bar{q}$. For a semilinear elliptic control problem in which the control enters the state equation on the right-hand side and not as a coefficient this issue has been studied in [12].

In the absence of inequality constraints the regularity of $\bar{q}$ is restricted only by the regularity of the domain Ω and by the regularity of the data $f,\ u_d,\ q_d$. However, the presence of control constraints leads to a stronger restriction on the regularity of $\bar{q}$, which often yields a reduction of the order of convergence of the finite element discretization.

We will prove the following convergence behavior, when discretizing the state variable by continuous cellwise (bi-/tri-) linear finite elements:

- $\mathcal{O}(h)$-convergence when discretizing the control variable by cellwise constants. This is a generalization of [13,18].
- $\mathcal{O}(h^{\frac{3}{2}})$-convergence when discretizing the control variable by (bi-/tri-)linear finite elements. This is a generalization of [12,18].
- $\mathcal{O}(h^2)$-convergence when discretizing the control variable by cellwise constants and applying a post-processing step. This is a generalization of [15,18].

To the knowledge of the authors, this is the first publication providing such estimates for the optimal control problem under consideration.

The paper is organized as follows: In Section 2 we formulate the optimal control problem in its functional analytic setting and recall some theoretical results concerning existence, uniqueness and regularity. In Section 3 we describe the finite element discretization of the optimal control problem. In Section 4 we prove some auxiliary estimates. In Section 5 we give explicit orders of convergence of the error $\|\bar{q}-\bar{q}_h\|_{L^2(\Omega)}$ and in Section 6 we confirm the theoretical results by some numerical examples.

2. Optimal control problem

In this section we briefly discuss the precise formulation of the optimization problem under consideration. Furthermore, we recall some theoretical results on existence, uniqueness, and regularity of optimal solutions as well as optimality conditions.

Let $\Omega \subset \mathbb{R}^d,\ d \in \{2,3\}$, be a convex polygonal domain. Here and in what follows, we employ the usual notion of Lebesgue, Sobolev, and Hölder spaces and we introduce the following notation: For inner products and norms on $L^2(\Omega)$ we use

$$(v,w)=(v,w)_{L^2(\Omega)} \quad \text{and} \quad \|v\|=\|v\|_{L^2(\Omega)}.$$

In addition, let $\|\cdot\|_{m,p}$ denote the norm on $W^{m,p}(\Omega)$ and $\|\cdot\|_p$ the norm on $L^p(\Omega)$ for $1\le m<\infty$, $m\in\mathbb{N}$, and $1\le p\le\infty$. Finally, let $C>0$ be a generic constant.

To formulate the optimal control problem we introduce the set Q_{ad} collecting the inequality constraints as

$$Q_{\mathrm{ad}}=\{q\in L^2(\Omega): a\le q\le b \text{ a.e. in } \Omega\},$$

where the bounds $a,b\in\mathbb{R}$ fulfill $0<a<b$. With the cost functional $J\colon Q_{\mathrm{ad}}\times H_0^1(\Omega)\to\mathbb{R}_0^+$ the weak formulation of the optimal control problem is given by:

$$\min_{q\in Q_{\mathrm{ad}},\ u\in H_0^1(\Omega)} J(q,u)=\frac{1}{2}\|u-u_d\|^2+\frac{\alpha}{2}\|q-q_d\|^2, \tag{2.1a}$$

subject to

$$(\nabla u,\nabla\varphi)+(qu,\varphi)=(f,\varphi)\quad \forall\varphi\in H_0^1(\Omega) \tag{2.1b}$$

for some $\alpha>0$.

Throughout this paper we make the following assumption:

Assumption 2.1. Let $u_d, f\in L^p(\Omega)$ for some $p>d$ and $q_d\in H^2(\Omega)\cap W^{1,\infty}(\Omega)$.

From standard arguments for elliptic equations we obtain the following proposition:

Proposition 2.2. *For every $q\in L^2(\Omega),\ q\ge 0$, the state equation (2.1b) admits a unique solution $u\in H_0^1(\Omega)$ and the following a priori estimate holds:*

$$\|\nabla u\| \le C\|f\|.$$

Moreover, for $q, p \in L^2(\Omega),\ q \ge 0,\ g \in H_0^1(\Omega)$ *let* $u \in H_0^1(\Omega)$ *be the unique solution of*

$$(\nabla u, \nabla \varphi) + (qu, \varphi) = (pg, \varphi) \quad \forall \varphi \in H_0^1(\Omega).$$

Then the following estimate holds:

$$\|\nabla u\| \le C\|\nabla g\|\|p\|.$$

Proposition 2.2 ensures the existence of a unique control-to-state mapping

$$S: W \to H_0^1(\Omega), \qquad q \mapsto S(q),$$

where $S(q)$ is the solution of (2.1b) and $W \supset Q_{\mathrm{ad}}$ is defined by

$$W = \{q \in L^\infty(\Omega) : \exists c > 0 : q > c > 0 \text{ a.e. in } \Omega\}.$$

By means of this mapping we introduce the reduced cost functional

$$j: W \to \mathbb{R}_0^+,$$
$$q \mapsto J(q, S(q)).$$

Hence, the optimal control problem (2.1) can be equivalently reformulated as

$$\min_{q \in Q_{\mathrm{ad}}} j(q).$$

From the form of the state equation (2.1b) we deduce the nonlinearity of the operator S and hence, the reduced functional j need not be convex, although the functional J is convex.

In the next proposition we show that the optimal control problem (2.1) admits a solution.

Proposition 2.3. *There exists a solution* $(\bar{q}, \bar{u}) \in L^2(\Omega) \times H_0^1(\Omega)$ *of problem* (2.1).

The proof follows standard techniques, we refer, e.g., to [19,20].

Since we cannot guarantee uniqueness of a solution of (2.1), we consider locally optimal solutions. Therefore, we use the following standard definition:

Definition 2.4 (*Local Solution*). A control $\bar{q} \in Q_{\mathrm{ad}}$ is called a local solution of (2.1), if there exists $\varepsilon > 0$, such that for all $q \in Q_{\mathrm{ad}}$ with $\|q - \bar{q}\| < \varepsilon$

$$j(q) \ge j(\bar{q})$$

holds.

From Proposition 2.3 we immediately obtain the existence of a local solution of the optimal control problem (2.1).

In what follows we need certain differentiation properties of the mappings S and j. Therefore, we introduce the following type of differentiability which we call Q-differentiability. Let X, Y, Z be Banach spaces.

Definition 2.5 (*Q-differentiability*). Let $Q \subset X$ be a convex set and $T: Q \to Y$. Then T is called to be Q-differentiable in $q \in Q$ with respect to Q, if there exists a mapping $T'_Q(q) \in \mathcal{L}(X, Y)$, such that for all $p \in Q$ holds

$$\frac{\|T(q + p - q) - T(q) - T'_Q(q)(p - q)\|_Y}{\|p - q\|_X} \to 0 \quad (\|p - q\|_X \to 0).$$

In the following we omit the index Q and write $T' = T'_Q$.

Remark 2.6. Let $Q \subset X$ be a convex set. Assume, $T: Q \to Y$ is Q-differentiable in $q \in Q$ with respect to Q and $G: Y \to Z$ Fréchet-differentiable in $T(q) \in Y$. Then $G \circ T$ is Q-differentiable in $q \in Q$ with respect to Q and the chain rule holds. Moreover, Q-differentiable functions satisfy the product rule.

By a straightforward calculation we verify the next proposition.

Lemma 2.7. *The control-to-state operator* $S: Q_{ad} \to H_0^1(\Omega)$ *is infinitely Q-differentiable in all* $q \in Q_{ad}$ *with respect to* Q_{ad}*. Moreover,* $j: Q_{ad} \to \mathbb{R}_0^+$ *is also at least three times Q-differentiable in all* $q \in Q_{ad}$ *with respect to* Q_{ad}*.*

Remark 2.8. The operators S and j are not Fréchet-differentiable with respect to the $L^2(\Omega)$–topology. However, since we do not want to use the so-called two-norm discrepancy, see also Remark 2.25, we need more than directional differentiability in all $q \in Q_{\mathrm{ad}}$ in the directions $p - q$ for $p \in Q_{\mathrm{ad}}$. Therefore, we have introduced the Q-differentiability which also implies directional differentiability in all $q \in Q_{\mathrm{ad}}$ in the directions $p - q$ for $p \in Q_{\mathrm{ad}}$.

Nevertheless, later we will use Fréchet-differentiability of S and j with respect to the $L^\infty(\Omega)$-topology to derive error estimates. Hence, we need the next lemma.

Lemma 2.9. *The operator S belongs to $C^\infty(W, H_0^1(\Omega))$ with respect to the $L^\infty(\Omega)$-topology and its derivatives have the following properties for all directions $p_1, p_2, p_3 \in L^\infty(\Omega)$:*

(i) *$S'(q)(p_1) \in H_0^1(\Omega)$ is the solution v of*

$$(\nabla v, \nabla\varphi) + (qv, \varphi) = -(p_1 S(q), \varphi) \quad \forall \varphi \in H_0^1(\Omega). \tag{2.2}$$

(ii) *$S''(q)(p_1, p_2) \in H_0^1(\Omega)$ is the solution w of*

$$(\nabla w, \nabla\varphi) + (qw, \varphi) = -(p_2 S'(q)(p_1), \varphi) - (p_1 S'(q)(p_2), \varphi) \quad \forall \varphi \in H_0^1(\Omega). \tag{2.3}$$

(iii) *$S'''(q)(p_1, p_2, p_3) \in H_0^1(\Omega)$ is the solution y of*

$$\begin{aligned}(\nabla y, \nabla\varphi) + (qy, \varphi) &= -(p_3 S''(q)(p_1, p_2), \varphi) - (p_2 S''(q)(p_1, p_3), \varphi) \\ &\quad - (p_1 S''(q)(p_2, p_3), \varphi) \quad \forall \varphi \in H_0^1(\Omega).\end{aligned}$$

Moreover, $j: W \to \mathbb{R}_0^+$ is at least three times Fréchet-differentiable.

The proof follows by a direct calculation.

Using directional-differentiability of j in $q \in Q_{\mathrm{ad}}$ in the directions $p - q$ for $p \in Q_{\mathrm{ad}}$ we can formulate the necessary optimality condition for a local solution:

Proposition 2.10. *Let $\bar{q} \in Q_{ad}$ be a local solution of (2.1). Then the following inequality holds:*

$$j'(\bar{q})(p - \bar{q}) \geq 0 \quad \forall p \in Q_{ad}. \tag{2.4}$$

For a standard proof we refer, e.g., to [19].

For given $q \in Q_{\mathrm{ad}}$ let $z \in H_0^1(\Omega)$ be the solution of the adjoint state equation

$$(\nabla\varphi, \nabla z) + (q\varphi, z) = (u - u_d, \varphi) \quad \forall \varphi \in H_0^1(\Omega) \tag{2.5}$$

with $u = S(q)$. Then the first directional derivative in $q \in Q_{\mathrm{ad}}$ in the direction $p - q$ for all $p \in Q_{\mathrm{ad}}$ of the reduced cost functional can be expressed as

$$j'(q)(p - q) = \alpha(q - q_d, p - q) - ((p - q)u, z). \tag{2.6}$$

For the solution of the adjoint equation (2.5) for given $q \in W \supset Q_{\mathrm{ad}}$ we can also introduce a control-to-adjoint-state operator:

Lemma 2.11. *There exists a unique operator*

$$Z: W \to H_0^1(\Omega), \qquad q \mapsto Z(q),$$

where $Z(q)$ is the solution of (2.5) and Z is infinitely Fréchet-differentiable.

The proof follows by the same arguments as we used to prove Proposition 2.2.

Using the projection operator $P_{[a,b]}$ defined on $L^2(\Omega)$ by

$$P_{[a,b]}(v)(x) = \min\{b, \max\{a, v(x)\}\} \text{ a.e. in } \Omega \text{ for } v \in L^2(\Omega)$$

every local solution $\bar{q} \in Q_{\mathrm{ad}}$ satisfying (2.4) fulfills

$$\bar{q} = P_{[a,b]}\left(\frac{1}{\alpha} S(\bar{q}) Z(\bar{q}) + q_d\right). \tag{2.7}$$

This can be verified by standard arguments, see, e.g., [20].

It is well known, that the projection $P_{[a,b]} : W^{1,\infty}(\Omega) \to W^{1,\infty}(\Omega)$ is continuous.

In what follows, we provide some stability estimates and give regularity results for the state, adjoint state, and control variable.

Proposition 2.12. *Let $g \in L^2(\Omega)$, $q \in L^\infty(\Omega)$, $q \geq 0$ and let $u \in H_0^1(\Omega)$ be the solution of*

$$(\nabla u, \nabla\varphi) + (qu, \varphi) = (g, \varphi) \quad \forall \varphi \in H_0^1(\Omega).$$

Then $u \in H^2(\Omega)$ and the following estimate holds

$$\|u\|_{2,2} \leq C(1 + \|q\|)\|g\|. \tag{2.8}$$

Proof. For a proof we refer to [21] and standard estimation techniques. □

Throughout this paper we make the following assumption.

Assumption 2.13. Let the solution of (2.1b) be in $W^{2,p}(\Omega)$ for some $p > d$ and all $q \in W$.

Remark 2.14. Due to the fact that Ω is a convex polygonal domain, for $d = 2$ there exists a constant $p_\Omega > 2$, such that for all $2 < p < p_\Omega$ Assumption 2.13 is fulfilled, see [21].

For $d = 3$ the domain Ω additionally has to satisfy a certain angle condition, then there exists a constant $p_\Omega > 3$, such that for all $3 < p < p_\Omega$ Assumption 2.13 is fulfilled, see [22].

Corollary 2.15. *Let $p \in L^2(\Omega)$, $q \in L^\infty(\Omega)$, $q \geq 0$, $g \in H^2(\Omega)$ and let $y \in H^1_0(\Omega)$ be the solution of*

$$(\nabla y, \nabla\varphi) + (qy, \varphi) = (pg, \varphi) \quad \forall\varphi \in H^1_0(\Omega).$$

Then the stability estimate

$$\|y\|_{2,2} \leq C(1 + \|q\|)\|g\|_{2,2}\|p\| \tag{2.9}$$

holds.

Proof. From the stability estimation (2.8) we derive

$$\|y\|_{2,2} \leq C(1 + \|q\|)\|pg\| \leq C(1 + \|q\|)\|g\|_\infty\|p\| \leq C(1 + \|q\|)\|g\|_{2,2}\|p\|. \quad \square$$

Remark 2.16. Let $q \in Q_{\mathrm{ad}}$ and $p \in L^\infty(\Omega)$. Then we deduce from Proposition 2.2 and Lemma 2.9 for $i \in \{1, 2, 3\}$ using Poincaré's inequality

$$\|S^{(i)}(q)(p^i)\| \leq C\|\nabla S^{(i)}(q)(p^i)\| \leq C\|\nabla S^{(i-1)}(q)(p^{i-1})\|\,\|p\|,$$

and since (2.9) we have

$$\|S^{(i)}(q)(p^i)\|_{2,2} \leq C(1 + \|q\|)\|S^{(i-1)}(q)(p^{i-1})\|_{2,2}\|p\|,$$

where $S^{(i)}(q)(p^i)$ denotes the ith-derivative of S at qi-times in the direction p and $S^{(0)}(q)(p^0) = S(q)$.

Accordingly, we have similar properties of Z, i.e., we have in particular

$$\begin{aligned}
&\|Z(q)\| \leq C\|\nabla Z(q)\| \leq C(\|f\| + \|u_d\|),\\
&\|Z'(q)(p)\| \leq C\|\nabla Z'(q)(p)\| \leq C(\|\nabla Z(q)\|\,\|p\| + \|S'(q)(p)\|),\\
&\|Z(q)\|_{2,2} \leq C(1 + \|q\|)(\|f\| + \|u_d\|).
\end{aligned}$$

Utilizing formulation (2.7) we obtain the following regularity result:

Lemma 2.17 (*Regularity*). *Let $\bar{q} \in Q_{ad}$ satisfy (2.7). Then the state $\bar{u} = S(\bar{q})$ and adjoint state $\bar{z} = Z(\bar{q})$ fulfill:*

$$\bar{u}, \bar{z} \in W^{2,p}(\Omega) \text{ for some } p > d$$

and hence,

$$\bar{q} \in W^{1,\infty}(\Omega).$$

Proof. From Assumption 2.13 we have $\bar{u}, \bar{z} \in W^{2,p}(\Omega)$ for some $p > d$ and hence, we can prove $\bar{u}\bar{z} \in W^{2,p}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ for some $p > d$. The projection $P_{[a,b]}: W^{1,\infty}(\Omega) \to W^{1,\infty}(\Omega)$ is continuous. Consequently, (2.7) implies the assertion. □

Hence, we can formulate some explicit representation of some a priori bounds for a local solution and its derivatives which we will need for the error estimates.

Remark 2.18. Applying Hölder's inequality and using (2.7) and Remark 2.16 we can estimate a local solution and its derivatives by the data:

$$\begin{aligned}
&\|\bar{q}\| \leq \frac{C}{\alpha}(\|f\|(\|f\| + \|u_d\|) + \|q_d\|),\\
&\|\nabla\bar{q}\| \leq \frac{C}{\alpha}(1 + \|\bar{q}\|)(\|f\|(\|f\| + \|u_d\|) + \|\nabla q_d\|),\\
&\|\nabla\bar{q}\|_\infty \leq \frac{1}{\alpha}(\|S(\bar{q})\|_\infty\|\nabla Z(\bar{q})\|_\infty + \|\nabla S(\bar{q})\|_\infty\|Z(\bar{q})\|_\infty + \|\nabla q_d\|_\infty).
\end{aligned}$$

On subsets of Ω a control $\bar{q} \in Q_{\mathrm{ad}}$ satisfying (2.4) might even have better regularity:

Remark 2.19. Let $\bar{q} \in Q_{ad}$ satisfy

$$\bar{q} = \frac{1}{\alpha} S(\bar{q}) Z(\bar{q}) + q_d$$

on a subset $\Omega' \subset \Omega$, then we have $\bar{q} \in H^2(\Omega')$ and by Remark 2.16 the following estimate is valid

$$\|\nabla^2 \bar{q}|_{\Omega'}\| \leq \frac{C}{\alpha}(1 + \|\bar{q}\|)^4 (\|f\|(\|f\| + \|u_d\|) + \|\nabla^2 q_d\|).$$

In the following we state a sufficient optimality condition. Since each local solution $\bar{q}$ is an element of the space W and because of the Fréchet-differentiability of j on W with respect to the $L^\infty(\Omega)$-topology the following assumption is well-formulated:

Assumption 2.20 (*Second-order Sufficient Optimality Condition*). Let $\bar{q}$ fullfill the necessary optimality condition (2.4). Then we assume, that there exists a constant $\gamma > 0$, such that

$$j''(\bar{q})(p, p) \geq \gamma \|p\|^2 \quad \forall p \in L^\infty(\Omega). \tag{2.10}$$

Remark 2.21. Assumption 2.20 is fulfilled for $\|S(\bar{q}) - u_d\|$ sufficiently small or α sufficiently large, since

$$\begin{aligned} j''(\bar{q})(p, p) &= (S'(\bar{q})(p), S'(\bar{q})(p)) + (S(\bar{q}) - u_d, S''(\bar{q})(p, p)) + \alpha(p, p) \\ &\geq (\alpha - C\|S(\bar{q}) - u_d\|)\|p\|^2. \end{aligned}$$

Usually, one cannot check the second-order sufficient optimality condition a priori. For a technique of numerical verification of second order sufficient optimality conditions we refer to [23].

To prove, that in a neighborhood of a local solution the second derivative of the reduced cost functional is also coercive, we need the next proposition.

Proposition 2.22. *The second derivative j'' of the reduced cost functional fulfills a Lipschitz-condition, i.e., there exists a constant $\hat{C} = C(\|f\|^2 + \|f\|\|u_d\|) > 0$, such that for all $p, q \in Q_{ad}$ and all $r \in L^\infty(\Omega)$*

$$|j''(q)(r, r) - j''(p)(r, r)| \leq \hat{C}\|q - p\|\|r\|^2$$

holds.

Proof. We have

$$\begin{aligned} j''(q)(r, r) - j''(p)(r, r) &= (S'(q)(r), S'(q)(r) - S'(p)(r)) + (S'(q)(r) - S'(p)(r), S'(p)(r)) \\ &\quad + (S(q) - u_d, S''(q)(r, r) - S''(p)(r, r)) + (S(q) - S(p), S''(p)(r, r)) \end{aligned}$$

and hence,

$$\begin{aligned} |j''(q)(r, r) - j''(p)(r, r)| &\leq \|S'(q)(r)\|\|S'(q)(r) - S'(p)(r)\| + \|S'(q)(r) - S'(p)(r)\|\|S'(p)(r)\| \\ &\quad + \|S(q) - u_d\|\|S''(q)(r, r) - S''(p)(r, r)\| + \|S(q) - S(p)\|\|S''(p)(r, r)\|. \end{aligned}$$

Applying the mean value theorem, Remark 2.16, and Proposition 2.2 we deduce the assertion. □

Lemma 2.23. *Let $\bar{q}$ be a local solution and the sufficient optimality condition (2.10) be true. There exists $\varepsilon > 0$, such that*

$$j''(q)(r, r) \geq \frac{\gamma}{2}\|r\|^2 \tag{2.11}$$

for all $r \in L^\infty(\Omega)$ and $q \in Q_{ad}$ with $\|q - \bar{q}\| \leq \varepsilon$.

Proof. Due to Assumption 2.20 and Proposition 2.22 we have

$$\begin{aligned} j''(q)(r, r) &= j''(\bar{q})(r, r) + (j''(q) - j''(\bar{q}))(r, r) \\ &\geq \gamma\|r\|^2 - C\varepsilon\|r\|^2 \\ &\geq \frac{\gamma}{2}\|r\|^2 \end{aligned}$$

for ε sufficiently small. □

If a given control satisfies the necessary and sufficient optimality conditions (2.4) and (2.10), then it is a local solution:

Theorem 2.24. *Let $\bar{q} \in Q_{ad}$ fulfill the necessary and sufficient optimality conditions (2.4) and (2.10). Then there are constants $\varepsilon, \sigma > 0$, such that*

$$j(q) \ge j(\bar{q}) + \sigma \|q - \bar{q}\|^2$$

for $q \in Q_{ad}$ and $\|q - \bar{q}\| \le \varepsilon$.

Proof. Using Q-differentiability, the proof follows by standard arguments, see, e.g., [20]. □

Remark 2.25. If we use the Fréchet-differentiability of S and j with respect to the $L^\infty(\Omega)$-topology and apply the theory of the so-called two-norm discrepancy, we get a slightly worse result:

$$j(q) \ge j(\bar{q}) + \sigma \|q - \bar{q}\|^2$$

for $q \in Q_{\text{ad}}$ and $\|q - \bar{q}\|_\infty \le \varepsilon$.

3. Discretization

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of meshes, whereas each mesh $\mathcal{T}_h$ is a triangulation of Ω in open quadrilaterals or hexahedrons, respectively with $h = \max\{h_K : K \in \mathcal{T}_h\}$, where $h_K = \text{diam}(K)$. We assume, that $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform, see, e.g., [24].

We define the conforming ansatz space for the state variable

$$V_h = \{v_h \in C(\bar{\Omega}) \; : v_h|_K \in \mathcal{Q}_1(K) \quad \forall K \in \mathcal{T}_h, \; v_h|_{\partial\Omega} = 0\},$$

where $\mathcal{Q}_1(K)$ consists of all shape functions obtained via (bi-/tri-)linear transformations of (bi-/tri-)linear functions defined on a reference cell $\hat{K} = [0, 1]^d$. For the discretization of the control space let $Q_h \subset L^2(\Omega)$ be a finite dimensional subspace and we define

$$Q_{\text{ad},h} = Q_{\text{ad}} \cap Q_h.$$

The space Q_h will either be the space of cellwise constant functions

$$Q_{h,0} = \{q_h \in L^2(\Omega) \; : \; v_h|_K = \text{const} \quad \forall K \in \mathcal{T}_h\}$$

or Q_h will be the space of continuous cellwise (bi-/tri-)linear finite elements similar to V_h:

$$Q_{h,1} = \{q_h \in C(\bar{\Omega}) \; : \; v_h|_K \in \mathcal{Q}_1(K) \quad \forall K \in \mathcal{T}_h\}.$$

The discrete optimization problem is formulated as follows:

$$\min_{q_h \in Q_{\text{ad},h}, u_h \in V_h} J(q_h, u_h) = \frac{1}{2}\|u_h - u_d\|^2 + \frac{\alpha}{2}\|q_h - q_d\|^2, \tag{3.1a}$$

subject to

$$(\nabla u_h, \nabla \varphi_h) + (q_h u_h, \varphi_h) = (f, \varphi_h) \quad \forall \varphi_h \in V_h. \tag{3.1b}$$

For this section and all following ones let the constant $C > 0$ be independent of the mesh parameter h.

As in Proposition 2.2 we have the following existence result and energy estimate:

Proposition 3.1. *For every $q \in L^2(\Omega)$, $q \ge 0$, the equation*

$$(\nabla u_h, \nabla \varphi_h) + (q u_h, \varphi_h) = (f, \varphi_h) \quad \forall \varphi_h \in V_h \tag{3.2}$$

admits a unique solution $u_h \in V_h$ and the following a priori estimate holds:

$$\|\nabla u_h\| \le C\|f\|.$$

Let $p \in L^2(\Omega)$, $g_h \in V_h$ and $u_h \in V_h$ be the solution of

$$(\nabla u_h, \nabla \varphi_h) + (q u_h, \varphi_h) = (p g_h, \varphi_h) \quad \forall \varphi_h \in V_h.$$

Then the estimate

$$\|\nabla u_h\| \le C\|\nabla g_h\|\|p\|$$

holds.

Moreover, as in the continuous case, we can introduce a discrete control-to-state operator:

Definition 3.2. There exists a unique discrete control-to-state operator S_h with

$$S_h : W \to V_h,$$
$$q \mapsto S_h(q),$$

where $S_h(q)$ is the solution of (3.2).

The operator is well-defined by Proposition 3.1.

Using this operator we introduce the discrete reduced cost functional

$$j_h: Q_{\text{ad}} \to \mathbb{R}_0^+, \qquad q \mapsto J(q, S_h(q))$$

and reformulate the discrete optimal control problem (3.1) as

$$\min_{q_h \in Q_{\text{ad},h}} j_h(q_h). \tag{3.3}$$

Further, we define a discrete local solution:

Definition 3.3 (*Discrete Local Solution*). A control $\bar{q}_h \in Q_{\text{ad},h}$ is called discrete local solution of (3.1), if there exists an $\varepsilon > 0$, such that for all $q_h \in Q_{\text{ad},h}$ with $\|q_h - \bar{q}_h\| < \varepsilon$

$$j_h(q_h) \geq j_h(\bar{q}_h)$$

holds.

As in the continuous case we obtain the existence of a local solution.

In the next lemma we summarize differentiability properties of the operators S_h and j_h, which we obtain in a similar way as in the continuous case.

Lemma 3.4. *The operator S_h belongs to $\in C^\infty(W, V_h)$ with respect to the $L^\infty(\Omega)$ –topology and its derivatives have the following properties for all directions $p_1, p_2, p_3 \in L^\infty(\Omega)$:*

(i) *$S_h'(q)(p_1) \in V_h$ is the solution v_h of*

$$(\nabla v_h, \nabla \varphi_h) + (q v_h, \varphi_h) = -(p_1 S_h(q), \varphi_h) \quad \forall \varphi_h \in V_h. \tag{3.4}$$

(ii) *$S_h''(q)(p_1, p_2) \in V_h$ is the solution w_h of*

$$(\nabla w_h, \nabla \varphi_h) + (q w_h, \varphi_h) = -(p_2 S_h'(q)(p_1), \varphi_h) - (p_1 S_h'(q)(p_2), \varphi_h) \quad \forall \varphi_h \in V_h. \tag{3.5}$$

(iii) *$S_h'''(q)(p_1, p_2, p_3) \in V_h$ is the solution y_h of*

$$\begin{aligned}(\nabla y_h, \nabla \varphi_h) + (q y_h, \varphi_h) &= -(p_3 S_h''(q)(p_1, p_2), \varphi_h) - (p_2 S_h''(q)(p_1, p_3), \varphi_h) \\ &\quad - (p_1 S_h''(q)(p_2, p_3), \varphi_h) \quad \forall \varphi_h \in V_h.\end{aligned}$$

Moreover, $j_h: W \to \mathbb{R}_0^+$ is at least three times Fréchet-differentiable with respect to the $L^\infty(\Omega)$-topology.

Remark 3.5. Applying Proposition 3.1, we have for $q \in Q_{\text{ad}}$, $p \in L^\infty(\Omega)$ and $i \in \{1, 2, 3\}$

$$\|S_h^{(i)}(q)(p^i)\| \leq C\|\nabla S_h^{(i)}(q)(p^i)\| \leq C\|\nabla S_h^{(i-1)}(q)(p^{i-1})\|\|p\|.$$

Thus, we can formulate the discrete necessary optimality condition for $\bar{q}_h \in Q_{\text{ad},h}$ as

$$j_h'(\bar{q}_h)(p_h - \bar{q}_h) \geq 0 \quad \forall p_h \in Q_{\text{ad},h},$$

where $j_h'(\bar{q}_h)(p_h - \bar{q}_h)$ is given by

$$j_h'(\bar{q}_h)(p_h - \bar{q}_h) = \alpha(\bar{q}_h - q_d, p_h - \bar{q}_h) - ((p_h - \bar{q}_h)\bar{u}_h, \bar{z}_h) \tag{3.6}$$

with $\bar{u}_h = S_h(\bar{q}_h)$ and the discrete adjoint solution $\bar{z}_h \in V_h$ of

$$(\nabla \varphi_h, \nabla \bar{z}_h) + (\bar{q}_h \varphi_h, \bar{z}_h) = (\bar{u}_h - u_d, \varphi_h) \quad \forall \varphi_h \in V_h. \tag{3.7}$$

For the solution of the discrete adjoint equation (3.7) we can also introduce a discrete control-to-adjoint-state operator:

Lemma 3.6. *There exists a unique infinitely Fréchet-differentiable operator $Z_h: Q_{ad} \to V_h$, $q \mapsto Z_h(q)$, where $Z_h(q)$ is the solution of (3.7).*

Remark 3.7. Applying Proposition 3.1 we have for $q \in Q_{\text{ad}}$, $p \in L^\infty(\Omega)$ the estimate

$$\|Z_h'(q)(p)\| \leq C\|\nabla Z_h'(q)(p)\| \leq C(\|\nabla Z_h(q)\|\|p\| + \|\nabla S_h'(q)(p)\|).$$

4. Auxiliary estimates

In this section we provide some auxiliary estimates for the error due to the discretization of the state and adjoint state variable. Furthermore, we deduce a discrete analogue to the coercivity condition (2.11).

4.1. Estimates for the discrete state and adjoint state variables

By standard finite element estimation techniques we obtain the following proposition.

Proposition 4.1. *Let $q \in L^\infty(\Omega)$, $q \geq 0$, $g \in L^2(\Omega)$ and assume, that $u \in H_0^1(\Omega)$ and $u_h \in V_h$ are the solutions of*

$$(\nabla u, \nabla\varphi) + (qu, \varphi) = (g, \varphi) \quad \forall \varphi \in H_0^1(\Omega) \quad \textit{and}$$
$$(\nabla u_h, \nabla\varphi_h) + (qu_h, \varphi_h) = (g, \varphi_h) \quad \forall \varphi_h \in V_h,$$

respectively. Then we have

$$\|\nabla(u - u_h)\| \leq C(1 + \|q\|)h\|\nabla^2 u\|,$$
$$\|u - u_h\| \leq C(1 + \|q\|)^2 h^2 \|\nabla^2 u\|.$$

Lemma 4.2. *Let $q \in L^\infty(\Omega)$, $q \geq 0$, $p \in L^2(\Omega)$, $g_1 \in H_0^1(\Omega) \cap H^2(\Omega)$, $g_2 \in H_0^1(\Omega)$ and assume, that $u \in H_0^1(\Omega)$ and $u_h \in V_h$ are the solutions of*

$$(\nabla u, \nabla\varphi) + (qu, \varphi) = (pg_1, \varphi) \quad \forall \varphi \in H_0^1(\Omega) \quad \textit{and}$$
$$(\nabla u_h, \nabla\varphi_h) + (qu_h, \varphi_h) = (pg_2, \varphi_h) \quad \forall \varphi_h \in V_h, \tag{4.1}$$

respectively. Then the following estimates hold:

$$\|\nabla(u - u_h)\| \leq C(1 + \|q\|)^2 h\|g_1\|_{2,2}\|p\| + C\|\nabla(g_1 - g_2)\|\|p\|,$$
$$\|u - u_h\| \leq C(1 + \|q\|)^3 h^2\|g_1\|_{2,2}\|p\| + C\|\nabla(g_1 - g_2)\|\|p\|.$$

Proof. Let $\hat{u}_h \in V_h$ be the solution of

$$(\nabla\hat{u}_h, \nabla\varphi_h) + (q\hat{u}_h, \varphi_h) = (pg_1, \varphi_h) \quad \forall \varphi_h \in V_h.$$

We subtract (4.1) and with $\varphi_h = \hat{u}_h - u_h$ we obtain using Hölder's inequality

$$\|\nabla(\hat{u}_h - u_h)\|^2 + (q(\hat{u}_h - u_h), \hat{u}_h - u_h) \leq \|p\|\|g_1 - g_2\|_4\|\hat{u}_h - u_h\|_4.$$

Applying the embedding theorem and Poincaré's inequality we get

$$\|\hat{u}_h - u_h\| \leq C\|\nabla(\hat{u}_h - u_h)\| \leq C\|p\|\|\nabla(g_1 - g_2)\|.$$

Consequently, using Proposition 4.1 and Corollary 2.15 we get

$$\begin{aligned}\|u - u_h\| &\leq \|u - \hat{u}_h\| + \|\hat{u}_h - u_h\| \\ &\leq C(1 + \|q\|)^2 h^2\|\nabla^2 u\| + C\|\nabla(g_1 - g_2)\|\|p\| \\ &\leq C(1 + \|q\|)^3 h^2\|g_1\|_{2,2}\|p\| + C\|\nabla(g_1 - g_2)\|\|p\|\end{aligned}$$

and accordingly,

$$\|\nabla(u - u_h)\| \leq C(1 + \|q\|)^2 h\|g_1\|_{2,2}\|p\| + C\|\nabla(g_1 - g_2)\|\|p\|. \quad \square$$

In what follows we summarize some estimates for the operators S, S_h and their derivatives.

Lemma 4.3. *Let $q, p \in Q_{ad}$ and $r \in L^\infty(\Omega)$. Then the following estimates hold for $m \in \{0, 1\}$:*

$$\|S(q) - S_h(q)\|_{m,2} \leq C(1 + \|q\|)^{3-m} h^{2-m}\|f\|, \tag{4.2}$$
$$\|S'(q)(r) - S_h'(q)(r)\| \leq C(1 + \|q\|)^4 h\|r\|\|f\|, \tag{4.3}$$
$$\|S''(q)(r, r) - S_h''(q)(r, r)\| \leq C(1 + \|q\|)^5 h\|r\|^2\|f\|, \tag{4.4}$$
$$\|S_h(q) - S_h(p)\| \leq C\|p - q\|\|f\|, \tag{4.5}$$
$$\|S_h(q)\|_\infty \leq C(1 + \|q\|)^3\|f\|. \tag{4.6}$$

Remark 4.4. With the standard estimation techniques we cannot prove quadratic convergence in (4.3) and (4.4) with r in the $L^2(\Omega)$-norm on the right-hand side of these error estimates. However, this fact does not influence the rate of convergence which we will later derive for the error between a continuous optimal control and the associate discrete one.

Proof. Assertion (4.2) follows directly from Proposition 4.1 and (2.8). Using Lemma 4.2, (2.2) and (3.4), we have

$$\|S'(q)(r) - S_h'(q)(r)\| \le C(1+\|q\|)^3 h^2 \|S(q)\|_{2,2}\|r\| + C\|\nabla(S(q) - S_h(q))\|\|r\|$$

and with (2.8) and (4.2)

$$\begin{aligned}\|S'(q)(r) - S_h'(q)(r)\| &\le C(1+\|q\|)^4 h^2 \|f\|\|r\| + C(1+\|q\|)^2 h\|\nabla^2 S(q)\|\|r\| \\ &\le C(1+\|q\|)^4 h\|f\|\|r\|.\end{aligned}$$

This implies (4.3). Accordingly, we deduce assertion (4.4) from (2.3) and (3.5). (4.5) is a direct consequence of the mean value theorem and Remark 3.5. Since the mesh is quasi-uniform, (4.6) follows with a standard estimate, see, e.g., [25,24]. □

For the operators Z and Z_h we can state similar estimates:

Lemma 4.5. *Let $q, p \in Q_{ad}$. Then the following estimates are valid for $m \in \{0, 1\}$:*

$$\|Z_h(q) - Z(q)\|_{m,2} \le C(1+\|q\|)^{3-m} h^{2-m}\|f\|,$$
$$\|Z_h(q) - Z_h(p)\| \le C\|p-q\|(\|f\| + \|u_d\|),$$
$$\|Z_h(q)\|_\infty \le C(1+\|q\|)^3(\|f\| + \|u_d\|).$$

4.2. Discrete coercivity

In this section we provide some auxiliary estimates and verify, that the second derivative of the discrete reduced cost functional is coercive in a neighborhood of a local solution, if the second-order optimality condition (2.10) is fulfilled for the continuous problem.

We start with some estimates for the first derivative of the reduced cost functional and its discrete analogue.

Lemma 4.6. *Let $q \in Q_{ad}$ and $r \in L^\infty(\Omega)$. Then the estimate*

$$|j'(q)(r) - j_h'(q)(r)| \le \hat{C} h^2 \|r\|$$

holds with $\hat{C} = C(1+\|q\|)^4(\|f\| + \|u_d\|)^2$.

Moreover, j_h' fulfills a Lipschitz condition, i.e., there exists a constant $\tilde{C} = C(\|f\|^2 + \|f\|\|u_d\| + \alpha) > 0$, such that for all $p, q \in Q_{ad}$ and all $r \in L^\infty(\Omega)$

$$|j_h'(q)(r) - j_h'(p)(r)| \le \tilde{C}\|q-p\|\|r\|.$$

Proof. By means of (2.6) and (3.6) we have

$$|j'(q)(r) - j_h'(q)(r)| \le |(r(S(q) - S_h(q)), Z(q))| + |(rS_h(q), Z(q) - Z_h(q))|.$$

Using the Lemmas 4.3 and 4.5 and Remark 2.16 we get with $C = C(\|f\|, \|u_d\|)$

$$\begin{aligned}|j'(q)(r) - j_h'(q)(r)| &\le C(1+\|q\|)(\|f\| + \|u_d\|)(\|S(q) - S_h(q)\| + \|Z(q) - Z_h(q)\|)\|r\| \\ &\le C(1+\|q\|)^4(\|f\| + \|u_d\|)^2 h^2\|r\|.\end{aligned}$$

This proves the first assertion.

The second assertion follows as in the proof of Proposition 2.22. □

In addition, we can show the coercivity of the second derivative of the discrete reduced cost functional in a neighborhood of a local solution.

Lemma 4.7. *Let $\bar{q}$ be a local solution of (2.1) and Assumption 2.20 be valid. Then there exists an $\varepsilon > 0$, such that for all $q \in Q_{ad}$ with $\|q - \bar{q}\| \le \varepsilon$ and all $r \in L^\infty(\Omega)$*

$$j_h''(q)(r, r) \ge \frac{\gamma}{4}\|r\|^2 \tag{4.7}$$

holds for h sufficiently small.

Proof. Using the explicit representations of j'' and j_h'' we have with $C = C(\|f\|, \|u_d\|)$ and (4.3) and (4.4), and Remark 2.18

$$\begin{aligned}|j''(q)(r,r) - j_h''(q)(r,r)| &\le C(1+\|q\|)^5 h\|r\|^2 \\ &\le C(1+\varepsilon+\|\bar{q}\|)^5 h\|r\|^2 \\ &\le \frac{\gamma}{4}\|r\|^2\end{aligned}$$

for h sufficiently small. Therefore, the assertion follows immediately from Lemma 2.23. □

5. Error estimates

In this section we prove the main results of this article, namely estimates for the error between a local solution $\bar{q}$ of the continuous optimal control problem (2.1) and an associate solution $\bar{q}_h$ of the discrete problem (3.1). Thereby, we will distinguish between different types of discretizations of the control variable.

We start with the formulation of an auxiliary problem for $\varepsilon > 0$, $h > 0$, to construct for a given local solution $\bar{q}$ an associate discrete one:

$$\min_{q_h \in U_\varepsilon^h(\bar{q})} j_h(q_h), \tag{5.1}$$

where $U_\varepsilon^h(\bar{q})$ is defined by

$$U_\varepsilon^h(\bar{q}) = \{q_h \in Q_{\mathrm{ad},h} : \|q_h - \bar{q}\| \le \varepsilon\} \subset L^\infty(\Omega).$$

In order to prove, that this auxiliary problem has a solution for h sufficiently small, we need the following proposition.

Proposition 5.1. *Let $\pi_h : L^2(\Omega) \to Q_{h,0}$ denote the L^2-projection operator defined by*

$$\pi_h q(x) = \frac{1}{|K|} \int_K q(\xi)\mathrm{d}\xi, \quad x \in K$$

for all $K \in \mathcal{T}_h$ and $q \in L^2(\Omega)$. Then $\pi_h Q_{ad} \subset Q_{ad} \cap Q_{h,0}$ and the estimate

$$\|\pi_h v - v\| \le ch\|\nabla v\| \tag{5.2}$$

holds for all $v \in H_0^1(\Omega)$. In addition, for all $q_h \in Q_{h,0}$, $p \in L^2(\Omega)$ we have

$$(q_h, \pi_h p - p) = 0.$$

Let $I_h : C(\bar{\Omega}) \to Q_{h,1}$ denote the usual nodal interpolation operator into the space $Q_{h,1}$ by pointwise setting

$$I_h g(x_i) = g(x_i)$$

for each node x_i of the triangulation $\mathcal{T}_h$ and $g \in C(\bar{\Omega})$. Then $I_h Q_{ad} \subset Q_{ad} \cap Q_{h,1}$ and the following estimate holds

$$\|I_h v - v\| \le ch\|\nabla v\|_\infty \tag{5.3}$$

for all $v \in W^{1,\infty}(\Omega)$.

A proof can be found, e.g., in [24,26].

Using this proposition we can state an existence assertion:

Lemma 5.2. *For all $\varepsilon > 0$ and $h > 0$ sufficiently small, the auxiliary problem (5.1) has a solution.*

Proof. Let

$$\hat{q}_h = \begin{cases} \pi_h \bar{q}, & \text{if } Q_h = Q_{h,0}, \\ I_h \bar{q}, & \text{if } Q_h = Q_{h,1}. \end{cases}$$

Then $\hat{q}_h \in Q_{\mathrm{ad},h}$ and for h small enough we have $\|\bar{q} - \hat{q}_h\| < \varepsilon$ and therefore, $\hat{q}_h \in U_h^\varepsilon(\bar{q})$. Hence, $U_h^\varepsilon(\bar{q})$ is not empty. For the further argumentation we refer to standard techniques as in Proposition 2.3. $\square$

Provided that ε and h are sufficiently small, the solution of (5.1) is unique:

Lemma 5.3. *Let $\varepsilon > 0$ be small enough, such that j_h'' satisfies (4.7) for $q \in U_\varepsilon^h(\bar{q})$, $p \in L^\infty(\Omega)$ and h sufficiently small. Then the auxiliary problem (5.1) has a unique solution.*

Proof. Let $\bar{q}_h, \bar{r}_h \in U_\varepsilon^h(\bar{q})$ be two global minima of j_h on $U_\varepsilon^h(\bar{q})$ with $\bar{r}_h \ne \bar{q}_h$ and $j_h(\bar{r}_h) = j_h(\bar{q}_h)$. Utilizing the necessary optimality condition and the coercivity we obtain for some $t \in [0, 1]$

$$\begin{aligned} j_h(\bar{r}_h) &= j_h(\bar{q}_h) + j_h'(\bar{q}_h)(\bar{r}_h - \bar{q}_h) + \frac{1}{2} j_h''(t\bar{r}_h + (1-t)\bar{q}_h)(\bar{r}_h - \bar{q}_h, \bar{r}_h - \bar{q}_h) \\ &\ge j_h(\bar{q}_h) + \frac{\gamma}{8}\|\bar{r}_h - \bar{q}_h\|^2 \end{aligned}$$

for h sufficiently small. As a result we get

$$0 = j_h(\bar{r}_h) - j_h(\bar{q}_h) \ge \frac{\gamma}{8}\|\bar{r}_h - \bar{q}_h\|^2 > 0$$

for h sufficiently small. This is a contradiction. $\square$

Under certain conditions the solution of (5.1) is also a discrete local solution of (3.1):

Lemma 5.4. *Let $\varepsilon > 0$ be small enough, such that j_h'' is coercive on $U_\varepsilon^h(\bar{q})$ for h sufficiently small. Moreover, let $\bar{q}_h^\varepsilon$ be a solution of (5.1) with $\bar{q}_h^\varepsilon \to \bar{q}$ for $h \to 0$ with respect to the $L^2(\Omega)$ –topology. Then $\bar{q}_h^\varepsilon$ is a local solution of (3.1) for h sufficiently small.*

Proof. The idea of the proof is taken from [12]. To prove, that $\bar{q}_h^\epsilon$ is a local solution of (3.1), we have to verify, that

$$j_h(q_h) \geq j_h(\bar{q}_h^\epsilon) \tag{5.4}$$

holds for all $q_h \in Q_{\mathrm{ad},h}$ with $\|q_h - \bar{q}_h^\epsilon\| \leq \frac{\epsilon}{2}$. By the definition of $\bar{q}_h^\epsilon$ we know (5.4) only for those $q_h \in Q_{\mathrm{ad},h}$ with $\|q_h - \bar{q}\| \leq \epsilon$. Let $q_h \in Q_{\mathrm{ad},h}$ satisfy $\|q_h - \bar{q}_h^\epsilon\| \leq \frac{\epsilon}{2}$. Then we have for h sufficiently small

$$\|q_h - \bar{q}\| \leq \|q_h - \bar{q}_h^\epsilon\| + \|\bar{q}_h^\epsilon - \bar{q}\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon.$$

This completes the proof. □

5.1. Cellwise constant discretization

In this section we discretize the control variable by cellwise constants, i.e.,

$$Q_h = Q_{h,0}$$

and show linear convergence with respect to h of the error $\|\bar{q}_h - \bar{q}\|$ for a sequence $(\bar{q}_h)_{h>0}$ of solutions of the discretized problem (3.1). In [13,18] this is proven for an elliptic and a parabolic problem, respectively with a linear control-to-state operator.

Theorem 5.5. *Let $\bar{q}$ be a local solution of (2.1) and Assumption 2.20 be valid. Then we can choose $\varepsilon > 0$ and $h > 0$ small enough, such that (5.1) has a unique solution denoted by $\bar{q}_h^\epsilon$ and the following estimate holds*

$$\|\bar{q} - \bar{q}_h^\varepsilon\| \leq C\frac{\alpha}{\sqrt{\gamma}} h\|\nabla q_d\| + \frac{\bar{C}}{\sqrt{\gamma}} h$$

for h sufficiently small and $\bar{C} = C(\|f\|, \|u_d\|, \|q_d\|, \|\nabla q_d\|, \alpha)$.

Proof. Let $\varepsilon > 0$ be small enough, such that

$$j''(q)(p,p) \geq \frac{\gamma}{2}\|p\|^2 \tag{5.5}$$

for all $q \in Q_{\mathrm{ad}}$ with $\|q - \bar{q}\| \leq \varepsilon$ and such that for h sufficiently small

$$j_h''(q_h)(p,p) \geq \frac{\gamma}{4}\|p\|^2 \tag{5.6}$$

for all $q_h \in U_\varepsilon^h(\bar{q}) = \{q_h \in Q_{\mathrm{ad},h} : \|q_h - \bar{q}\| \leq \varepsilon\}$ and $p \in L^\infty(\Omega)$. This is possible, see Lemmas 2.23 and 4.7. With this ε we consider (5.1) and formulate another auxiliary problem

$$\min_{q_h \in U_\varepsilon^h(\bar{q})} j(q_h), \tag{5.7}$$

where we only discretize the control variable. For h sufficiently small (5.1) and (5.7) have unique solutions. This is a consequence of the Lemmas 5.2 and 5.3, which are also valid, if we replace j_h by j. We denote the solutions of (5.1) and (5.7) by $\bar{q}_h^\varepsilon$ and $\hat{q}_h^\varepsilon$, respectively.

To derive an error estimate, we split the error

$$\|\bar{q} - \bar{q}_h^\varepsilon\| \leq \|\bar{q} - \hat{q}_h^\varepsilon\| + \|\hat{q}_h^\varepsilon - \bar{q}_h^\varepsilon\|. \tag{5.8}$$

By (5.5) we have for a $t \in [0, 1]$ with $\xi = t\bar{q} + (1-t)\hat{q}_h^\varepsilon$ and h sufficiently small

$$\begin{aligned}\frac{\gamma}{2}\|\bar{q} - \hat{q}_h^\varepsilon\|^2 &\leq j''(\xi)(\bar{q} - \hat{q}_h^\varepsilon, \bar{q} - \hat{q}_h^\varepsilon)\\ &= j'(\bar{q})(\bar{q} - \hat{q}_h^\varepsilon) - j'(\hat{q}_h^\varepsilon)(\bar{q} - \hat{q}_h^\varepsilon)\\ &= j'(\bar{q})(\bar{q} - \hat{q}_h^\varepsilon) - j'(\hat{q}_h^\varepsilon)(\bar{q} - \pi_h\bar{q}) - j'(\hat{q}_h^\varepsilon)(\pi_h\bar{q} - \hat{q}_h^\varepsilon).\end{aligned}$$

The necessary optimality conditions imply for h sufficiently small

$$j'(\bar{q})(\bar{q} - \hat{q}_h^\varepsilon) \leq 0 \quad \text{and} \quad -j'(\hat{q}_h^\varepsilon)(\pi_h\bar{q} - \hat{q}_h^\varepsilon) \leq 0,$$

and hence, we get with the properties of π_h and Young's inequality

$$\begin{aligned}\frac{\gamma}{2}\|\bar q-\hat q_h^\varepsilon\|^2 &\le -j'(\hat q_h^\varepsilon)(\bar q-\pi_h\bar q)\\ &= -(\alpha(\hat q_h^\varepsilon-q_d)-S(\hat q_h^\varepsilon)Z(\hat q_h^\varepsilon),\bar q-\pi_h\bar q)\\ &= -(\alpha(\pi_h q_d-q_d),\bar q-\pi_h\bar q)+(S(\hat q_h^\varepsilon)Z(\hat q_h^\varepsilon)-\pi_h(S(\hat q_h^\varepsilon)Z(\hat q_h^\varepsilon)),\bar q-\pi_h\bar q)\\ &\le \frac{\alpha^2}{2}\|q_d-\pi_h q_d\|^2+\frac12\|S(\hat q_h^\varepsilon)Z(\hat q_h^\varepsilon)-\pi_h(S(\hat q_h^\varepsilon)Z(\hat q_h^\varepsilon))\|^2+\|\bar q-\pi_h\bar q\|^2.\end{aligned}$$

Therefore, we have

$$\|\bar q-\hat q_h^\varepsilon\|\le C\frac{\alpha}{\sqrt\gamma}\|q_d-\pi_h q_d\|+\frac{C}{\sqrt\gamma}\|S(\hat q_h^\varepsilon)Z(\hat q_h^\varepsilon)-\pi_h(S(\hat q_h^\varepsilon)Z(\hat q_h^\varepsilon))\|+\frac{C}{\sqrt\gamma}\|\bar q-\pi_h\bar q\|.$$

Applying (5.2) we obtain

$$\|\bar q-\hat q_h^\varepsilon\|\le C\frac{\alpha}{\sqrt\gamma}h\|\nabla q_d\|+\frac{C}{\sqrt\gamma}h\|\nabla(S(\hat q_h^\varepsilon)Z(\hat q_h^\varepsilon))\|+\frac{C}{\sqrt\gamma}h\|\nabla\bar q\| \tag{5.9}$$

and further, we have with Remark 2.16

$$\begin{aligned}\|\nabla(S(\hat q_h^\varepsilon)Z(\hat q_h^\varepsilon))\| &\le \|\nabla S(\hat q_h^\varepsilon)\|\|Z(\hat q_h^\varepsilon)\|_\infty+\|S(\hat q_h^\varepsilon)\|_\infty\|\nabla Z(\hat q_h^\varepsilon)\|\\ &\le C(1+\varepsilon+\|\bar q\|)\|f\|(\|f\|+\|u_d\|).\end{aligned}\tag{5.10}$$

Summarizing, we deduce from (5.9) and (5.10)

$$\|\bar q-\hat q_h^\varepsilon\|\le C\frac{\alpha}{\sqrt\gamma}h\|\nabla q_d\|+\frac{C(1+\varepsilon+\|\bar q\|)}{\sqrt\gamma}h\|f\|(\|f\|+\|u_d\|)+\frac{C}{\sqrt\gamma}h\|\nabla\bar q\|. \tag{5.11}$$

To estimate the second term in (5.8), we exploit the necessary optimality conditions leading to the following relation for all $r_h\in U_\varepsilon^h(\bar q)$

$$j_h'(\bar q_h^\varepsilon)(\bar q_h^\varepsilon-r_h)\le 0\le j'(\hat q_h^\varepsilon)(r_h-\hat q_h^\varepsilon).$$

Hence, we obtain with (5.6) the following estimate for $\xi=t\bar q_h^\varepsilon+(1-t)\hat q_h^\varepsilon$ with a $t\in[0,1]$ and h sufficiently small:

$$\begin{aligned}\frac{\gamma}{4}\|\bar q_h^\varepsilon-\hat q_h^\varepsilon\|^2 &\le j_h''(\xi)(\bar q_h^\varepsilon-\hat q_h^\varepsilon,\bar q_h^\varepsilon-\hat q_h^\varepsilon)\\ &= j_h'(\bar q_h^\varepsilon)(\bar q_h^\varepsilon-\hat q_h^\varepsilon)-j_h'(\hat q_h^\varepsilon)(\bar q_h^\varepsilon-\hat q_h^\varepsilon)\\ &\le j'(\hat q_h^\varepsilon)(\bar q_h^\varepsilon-\hat q_h^\varepsilon)-j_h'(\hat q_h^\varepsilon)(\bar q_h^\varepsilon-\hat q_h^\varepsilon)\\ &\le C(1+\varepsilon+\|\bar q\|)^4(\|f\|+\|u_d\|)^2h^2\|\bar q_h^\varepsilon-\hat q_h^\varepsilon\|.\end{aligned}\tag{5.12}$$

The last step follows from Lemma 4.6 and since $(1+\|\hat q_h^\varepsilon\|)\le(1+\varepsilon+\|\bar q\|)$.

Using Remark 2.18 and inserting (5.11) and (5.12) in (5.8) yields the assertion. □

This theorem implies the following result:

Corollary 5.6. *Let $\bar q$ be a local solution of (2.1) and Assumption 2.20 be valid. Then for a $h_0>0$ there exists a sequence $(\bar q_h)_{0<h<h_0}$ of discrete solutions of (3.1), such that the following estimate holds*

$$\|\bar q_h-\bar q\|\le C\frac{\alpha}{\sqrt\gamma}h\|\nabla q_d\|+\frac{\bar C}{\sqrt\gamma}h$$

with $\bar C=C(\|f\|,\|u_d\|,\|q_d\|,\|\nabla q_d\|,\alpha)$.

Proof. From the Lemmas 5.3 and 5.4 we derive, that we can choose $\varepsilon>0$ small enough, such that for $h>0$ sufficiently small the solution $\bar q_h^\varepsilon$ of (5.1) is a local solution of (3.1). Hence, the assertion follows from Theorem 5.5. □

5.2. Cellwise linear discretization

This section is devoted to the error analysis for the discretization of the control variable by cellwise (bi-/tri-)linear functions, i.e., we choose

$$Q_h=Q_{h,1}.$$

The analysis of this section and the following one is based on an assumption on the structure of the active sets. Let $\bar{q} \in Q_{\text{ad}}$ be a local solution. Then we group the cells K of the mesh $\mathcal{T}_h$ depending on the value of $\bar{q}_K$ on K into the three sets $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^2 \cup \mathcal{T}_h^3$ with $\mathcal{T}_h^i \cap \mathcal{T}_h^j = \emptyset$ for $i \neq j$. The sets are

$$\mathcal{T}_h^1 = \{K \in \mathcal{T}_h : \bar{q}(x) = a \text{ or } \bar{q}(x) = b \text{ for all } x \in K\},$$
$$\mathcal{T}_h^2 = \{K \in \mathcal{T}_h : a < \bar{q} < b \text{ for all } x \in K\},$$
$$\mathcal{T}_h^3 = \mathcal{T}_h \setminus (\mathcal{T}_h^1 \cup \mathcal{T}_h^2).$$

Assumption 5.7. We assume that there exists a positive constant C independent of h, such that

$$\sum_{K \in \mathcal{T}_h^3} |K| \leq Ch.$$

Remark 5.8. A similar assumption is used in [15,11,18]. This assumption is valid if the boundary of the level sets

$$\{x \in \Omega : \bar{q}(x) = a\} \quad \text{and} \quad \{x \in \Omega : \bar{q}(x) = b\}$$

consists of a finite number of rectifiable curves.

Let I_h be defined as in Proposition 5.1. Then we can state the following theorem.

Theorem 5.9. *Let $\bar{q}$ be a local solution of (2.1) and Assumption 2.20 be valid. Then we can choose a $h_0 > 0$, such that there exists a sequence $(\bar{q}_h)_{0<h<h_0}$ of discrete local solutions of (3.1) and the following estimate holds*

$$\|\bar{q} - \bar{q}_h\| \leq \left(1 + \frac{C}{\gamma}\right) \|I_h\bar{q} - \bar{q}\| + \frac{C}{\sqrt{\gamma}}\sqrt{j'(\bar{q})(I_h\bar{q} - \bar{q})} + \frac{\bar{C}}{\gamma}h^2 \tag{5.13}$$

with the constants

$$C = C(\|f\|, \|u_d\|, \alpha) \quad \textit{and} \quad \bar{C} = C(\|f\|, \|u_d\|, \|q_d\|, \alpha).$$

Proof. Let $\varepsilon > 0$ be small enough, such that for $h > 0$ sufficiently small j_h'' satisfies (4.7) for all $q \in Q_{\text{ad}}$ with $\|q - \bar{q}\| \leq \varepsilon$ and $p \in L^\infty(\Omega)$. Then for h sufficiently small the auxiliary problem (5.1) has a unique solution. We denote this solution by $\bar{q}_h$.

To derive the estimate (5.13) we split the error

$$\|\bar{q} - \bar{q}_h\| \leq \|\bar{q} - I_h\bar{q}\| + \|I_h\bar{q} - \bar{q}_h\| \tag{5.14}$$

and estimate the term $\|I_h\bar{q} - \bar{q}_h\|$. Due to the necessary optimality condition and since $I_h\bar{q} \in Q_{\text{ad},h}$, we have

$$-j_h'(\bar{q}_h)(I_h\bar{q} - \bar{q}_h) \leq 0 \leq -j'(\bar{q})(\bar{q} - \bar{q}_h).$$

Applying the coercivity of j_h'', we obtain for $\xi = t\bar{q}_h + (1-t)I_h\bar{q}$ for a $t \in [0, 1]$ and h sufficiently small

$$\begin{aligned}
\frac{\gamma}{4}\|I_h\bar{q} - \bar{q}_h\|^2 &\leq j_h''(\xi)(I_h\bar{q} - \bar{q}_h, I_h\bar{q} - \bar{q}_h) \\
&\leq j_h'(I_h\bar{q})(I_h\bar{q} - \bar{q}_h) - j_h'(\bar{q}_h)(I_h\bar{q} - \bar{q}_h) \\
&\leq j_h'(I_h\bar{q})(I_h\bar{q} - \bar{q}_h) - j'(\bar{q})(\bar{q} - \bar{q}_h) \\
&= j_h'(I_h\bar{q})(I_h\bar{q} - \bar{q}_h) - j_h'(\bar{q})(I_h\bar{q} - \bar{q}_h) + j_h'(\bar{q})(I_h\bar{q} - \bar{q}_h) - j'(\bar{q})(I_h\bar{q} - \bar{q}_h) + j'(\bar{q})(I_h\bar{q} - \bar{q}).
\end{aligned} \tag{5.15}$$

From Lemma 4.6 we deduce for h sufficiently small

$$|(j_h'(I_h\bar{q}) - j_h'(\bar{q}))(I_h\bar{q} - \bar{q}_h)| \leq C(\|f\|^2 + \|f\|\|u_d\| + \alpha)\|I_h\bar{q} - \bar{q}\|\|I_h\bar{q} - \bar{q}_h\|$$

and

$$|(j_h'(\bar{q}) - j'(\bar{q}))(I_h\bar{q} - \bar{q}_h)| \leq C(1 + \|\bar{q}\|)^4(\|f\| + \|u_d\|)^2 h^2 \|I_h\bar{q} - \bar{q}_h\|.$$

Applying these inequalities to the right-hand side of (5.15) leads to

$$\|I_h\bar{q} - \bar{q}_h\| \leq \frac{C}{\gamma}(\|f\|^2 + \|f\|\|u_d\| + \alpha)\|I_h\bar{q} - \bar{q}\| + \frac{C}{\gamma}(1 + \|\bar{q}\|)^4(\|f\| + \|u_d\|)^2 h^2 + \frac{C}{\sqrt{\gamma}}\sqrt{j'(\bar{q})(I_h\bar{q} - \bar{q})}$$

for h sufficiently small. Inserting this estimate into (5.14) together with Remark 2.18 we have proved the estimate (5.13) and hence, convergence of a solution $\bar{q}_h$ of (5.1) to $\bar{q}$. Therefore, we obtain by Lemma 5.4, that $\bar{q}_h$ is also a local solution of (3.1), which completes the proof. $\square$

Corollary 5.10. *Let $\bar{q}$ be a local solution of (2.1) and the Assumptions 2.20 and 5.7 be valid. Then there exists a $h_0 > 0$, such that there exists a sequence $(\bar{q}_h)_{0<h<h_0}$ of discrete local solutions of (3.1) and the following estimate holds*

$$\|\bar{q} - \bar{q}_h\| \leq \frac{C}{\gamma} h^{\frac{3}{2}}$$

with the constant

$$\bar{C} = C(\|f\|, \|u_d\|, \|q_d\|, \|\nabla^2 q_d\|, \|\nabla S(\bar{q})\|_\infty, \|\nabla Z(\bar{q})\|_\infty, \|\nabla q_d\|_\infty, \alpha).$$

Proof. As in [11,18] we prove, that

$$j'(\bar{q})(I_h\bar{q} - \bar{q}_h) \leq Ch^3 \|\nabla \bar{q}\|_\infty^2,$$
$$\|I_h\bar{q} - \bar{q}\| \leq Ch^2 \|\nabla^2 \bar{q}\|_{L^2(\mathcal{T}_h^2)} + Ch^{\frac{3}{2}} \|\nabla \bar{q}\|_\infty.$$

Together with Theorem 5.9 and Remark 2.19 we obtain the assertion. □

Remark 5.11. If we assume, $q_d \in W^{2,p}(\Omega)$ for some $p > d$, then in case of inactive control constraints, i.e., $a < \bar{q} < b$, we can prove convergence of order $\mathcal{O}(h^2)$ in the control.

5.3. Post-processing strategy

In this section, we extend the post-processing techniques initially proposed in [15] for a linear-quadratic optimal control problem to the optimal control problem under consideration.

As described in Section 5.1, we discretize the control variable by piecewise constants. But here, we will prove quadratic order of convergence by employing a post-processing step.

In what follows we use the operator R_h defined for functions $g \in C(\bar{\Omega})$ cellwise by

$$R_h g|_K = g(S_K), \quad K \in \mathcal{T}_h,$$

where S_K denotes the barycenter of the cell K.

Lemma 5.12. *Let $K \in \mathcal{T}_h$ be a given cell. Then we have for $g \in H^2(K)$*

$$\left| \int_K (g(x) - (R_h g)(x)) \mathrm{d}x \right| \leq Ch^2 |K|^{\frac{1}{2}} \|\nabla^2 g\|_{L^2(K)}, \tag{5.16}$$

and for $g \in W^{1,\infty}(K)$

$$\|g - R_h g\|_{L^\infty(K)} \leq Ch \|\nabla g\|_{L^\infty(K)}. \tag{5.17}$$

Proof. The proof is done by standard arguments using the Bramble–Hilbert Lemma, see [15] for details. □

The proofs of the next two lemmas are similar to lemmas in [18].

Lemma 5.13. *Let $\bar{q}$ be a local solution of (2.1) and $\bar{q}_h$ be an arbitrary local solution of (3.1). Then the following estimate holds*

$$0 \leq (\alpha R_h \bar{q} + R_h(z(\bar{q})u(\bar{q})) + R_h q_d, \bar{q}_h - R_h \bar{q}).$$

Lemma 5.14. *For every function $v \in H^2(\Omega)$ and every cellwise constant function $p_h \in Q_h$ the estimate*

$$(p_h, v - R_h v) \leq Ch^2 \|p_h\| \|\nabla^2 v\|$$

holds.

In the next step, we estimate the error $\|R_h\bar{q} - \bar{q}_h\|$. To this end, we need the following two lemmas.

Lemma 5.15. *Let $\bar{q} \in Q_{ad}$ be a local solution of (2.1) and Assumption 5.7 be valid. Furthermore, let $v_h, w_h \in V_h$. Then the following estimate holds for an arbitrary $r \geq 3$:*

$$(v_h w_h, \bar{q} - R_h \bar{q}) \leq Ch^2(\|w_h\|_{1,r} + \|w_h\|_\infty)\|\nabla v_h\| + Ch^2 \|w_h\|_\infty \|v_h\|_\infty$$

with the constant

$$C = C(\|f\|, \|u_d\|, \|q_d\|, \|\nabla q_d\|, \|\nabla^2 q_d\|, \|\nabla S(\bar{q})\|_\infty, \|\nabla Z(\bar{q})\|_\infty, \|\nabla q_d\|_\infty, \alpha).$$

The proof is given in the Appendix.

Before we formulate the next lemma we recall an estimate from [26, Theorem (8.1.11)]. For $2 < s < \infty$ and a constant $\tilde{C} \geq 0$ there holds

$$\|S_h(R_h\bar{q})\|_{1,s} \leq \tilde{C}\|S(R_h\bar{q})\|_{1,s}. \tag{5.18}$$

Lemma 5.16. *Let $\bar{q} \in Q_{ad}$ be a local solution of (2.1) and Assumption 5.7 be valid. Then the estimates*

$$\|S_h(R_h\bar{q}) - S_h(\bar{q})\| \leq Ch^2,$$
$$\|Z_h(R_h\bar{q}) - Z_h(\bar{q})\| \leq Ch^2$$

hold with the constant

$$C = C(\tilde{C}, \|f\|, \|u_d\|, \|q_d\|, \|\nabla q_d\|, \|\nabla^2 q_d\|, \|\nabla S(\bar{q})\|_\infty, \|\nabla Z(\bar{q})\|_\infty, \|\nabla q_d\|_\infty, \alpha)$$

and h sufficiently small.

Proof. At first let $e = S_h(R_h\bar{q}) - S_h(\bar{q})$. Let y be the solution of

$$(\nabla y, \nabla\varphi) + (\bar{q}y, \varphi) = (e, \varphi) \quad \forall\varphi \in H_0^1(\Omega).$$

Then we have

$$\begin{aligned}\|e\|^2 &= (\nabla e, \nabla y) + (\bar{q}y, e)\\ &= (\nabla e, \nabla(y - I_h y)) + (\bar{q}e, y - I_h y) + (\nabla e, \nabla I_h y) + (\bar{q}e, I_h y).\end{aligned} \tag{5.19}$$

Since

$$(\nabla S_h(R_h\bar{q}), \nabla\varphi_h) + (R_h\bar{q} \cdot S_h(R_h\bar{q}), \varphi_h) = (f, \varphi_h),$$
$$(\nabla S_h(\bar{q}), \nabla\varphi_h) + (\bar{q}S_h(\bar{q}), \varphi_h) = (f, \varphi_h),$$

we have

$$0 = (\nabla e, \nabla\varphi_h) + (\bar{q}(S_h(R_h\bar{q}) - S_h(\bar{q})), \varphi_h) + ((R_h(\bar{q}) - \bar{q})S_h(R_h\bar{q}), \varphi_h). \tag{5.20}$$

From (5.18) we obtain for $2 < s \leq 6$ using the embedding theorem and Remark 2.16

$$\begin{aligned}\|S_h(R_h\bar{q})\|_{1,s} &\leq \tilde{C}\|S(R_h\bar{q})\|_{1,s}\\ &\leq C\tilde{C}(1 + \|R_h\bar{q}\|)\|f\|\\ &\leq C\tilde{C}(1 + \|\bar{q}\|)\|f\|\\ &\leq C(\tilde{C}, \|f\|, \|u_d\|, \|q_d\|, \alpha)\end{aligned}$$

for h sufficiently small. Hence, setting $v_h = I_h y$ and $w_h = S_h(R_h(\bar{q}))$ in Lemma 5.15 we obtain from Lemma 5.15 applying Lemma 4.3, Remark 2.18, and (5.20) for $3 \leq r \leq 6$

$$\begin{aligned}(\nabla e, \nabla I_h y) + (\bar{q}e, I_h y) &= -((R_h(\bar{q}) - \bar{q})S_h(R_h\bar{q}), I_h y)\\ &\leq Ch^2(\|S_h(R_h\bar{q})\|_{1,r} + \|S_h(R_h\bar{q})\|_\infty)\|\nabla I_h y\| + Ch^2\|S_h(R_h\bar{q})\|_\infty\|I_h y\|_\infty\\ &\leq Ch^2\|\nabla I_h y\| + Ch^2\|I_h y\|_\infty\end{aligned} \tag{5.21}$$

with

$$C = C(\tilde{C}, \|f\|, \|u_d\|, \|q_d\|, \|\nabla q_d\|, \|\nabla^2 q_d\|, \|\nabla S(\bar{q})\|_\infty, \|\nabla Z(\bar{q})\|_\infty, \|\nabla q_d\|_\infty, \alpha)$$

and h sufficiently small. In addition, we have

$$\|I_h y\|_\infty \leq \|I_h y - y\|_\infty + \|y\|_\infty \leq Ch\|\nabla^2 y\| + \|y\|_\infty \leq (1 + \|\bar{q}\|)\|e\|. \tag{5.22}$$

Inserting (5.21) and (5.22) in (5.19) and using Lipschitz continuity of S_h and Remark 2.16 we get

$$\begin{aligned}\|e\|^2 &\leq \|\nabla(S_h(R_h\bar{q}) - S_h(\bar{q}))\|\|\nabla(y - I_h y)\| + C\|\bar{q}\|\|\nabla(S_h(R_h\bar{q}) - S_h(\bar{q}))\|\|\nabla(y - I_h y)\| + Ch^2\|e\|\\ &\leq C\|R_h\bar{q} - \bar{q}\|\|\nabla(y - I_h y)\| + C\|\bar{q}\|\|R_h\bar{q} - \bar{q}\|\|\nabla(y - I_h y)\| + Ch^2\|e\|\\ &\leq C(1 + \|\bar{q}\|)h^2\|\nabla\bar{q}\|\|e\| + C(1 + \|\bar{q}\|)\|\bar{q}\|h^2\|\nabla\bar{q}\|\|e\| + Ch^2\|e\|\\ &\leq Ch^2\|e\|.\end{aligned}$$

For $e = Z_h(R_h\bar{q}) - Z_h(\bar{q})$ we have instead of (5.20)

$$0 = (\nabla e, \nabla\varphi_h) + (\bar{q}(Z_h(R_h\bar{q}) - Z_h(\bar{q})), \varphi_h) + ((R_h(\bar{q}) - \bar{q})Z_h(R_h\bar{q}), \varphi_h) + (S_h(\bar{q}) - S_h(R_h\bar{q}), \varphi_h)$$

and we can argue in the same way as above using

$$\begin{aligned}(S_h(\bar q) - S_h(R_h\bar q), I_h y) &\le C\|S_h(\bar q) - S_h(R_h\bar q)\|\|e\| \\ &\le Ch^2\|e\|. \quad \square\end{aligned}$$

Lemma 5.17. *Let $\bar q$ be a local solution of (2.1) and the Assumptions 2.20 and 5.7 be valid. Then there exists a $h_0 > 0$, such that there exists a sequence $(\bar q_h)_{0<h<h_0}$ of discrete local solutions of (3.1) with $\|\bar q - \bar q_h\| = \mathcal{O}(h)$ and the estimate*

$$\|R_h\bar q - \bar q_h\| \le Ch^2 \tag{5.23}$$

holds with the constant

$$C = C(\tilde C, \|f\|, \|u_d\|, \|q_d\|, \|\nabla q_d\|, \|\nabla^2 q_d\|, \|\nabla S(\bar q)\|_\infty, \|\nabla Z(\bar q)\|_\infty, \|\nabla q_d\|_\infty, \alpha).$$

The proof is given in the Appendix.

Let $\bar q \in Q_{\mathrm{ad}}$ be a local solution of (2.1) and $\bar q_h$ the corresponding discrete local solution of (3.1) with $\|\bar q - \bar q_h\| = \mathcal{O}(h)$, see Corollary 5.6. Then a better approximation is constructed by a post-processing step making use of the projection operator:

$$\tilde q_h = P_{[a,b]}\left(\frac{1}{\alpha}S_h(\bar q_h)Z_h(\bar q_h) + I_h q_d\right). \tag{5.24}$$

Here, I_h denotes the operator defined in Section 5.1. Thus, we can formulate the main result of this section:

Theorem 5.18. *Let $\bar q$ be a local solution of (2.1) and the Assumptions 2.20 and 5.7 be valid. Then we can choose a $h_0 > 0$, such that there exists a sequence $(\bar q_h)_{0<h<h_0}$ of discrete local solutions of (3.1) and the following estimate holds*

$$\|\bar q - \tilde q_h\| \le Ch^2$$

with the constant

$$C = C(\tilde C, \|f\|, \|u_d\|, \|q_d\|, \|\nabla q_d\|, \|\nabla^2 q_d\|, \|\nabla S(\bar q)\|_\infty, \|\nabla Z(\bar q)\|_\infty, \|\nabla q_d\|_\infty, \alpha)$$

and where $\tilde q_h$ is defined by (5.24).

Proof. By the optimality condition (2.7) and the definition of $\tilde q_h$ we have

$$\|\bar q - \tilde q_h\| = \left\|P_{[a,b]}\left(\frac{1}{\alpha}Z(\bar q)S(\bar q) + q_d\right) - P_{[a,b]}\left(\frac{1}{\alpha}Z_h(\bar q_h)S_h(\bar q_h) + I_h q_d\right)\right\|.$$

Further, using the Lipschitz continuity of $P_{[a,b]}$ on $L^2(\Omega)$ and Remark 2.16, we have

$$\|\bar q - \tilde q_h\| \le C(\|S_h(\bar q_h) - S(\bar q)\| + \|Z_h(\bar q_h) - Z(\bar q)\| + \|q_d - I_h q_d\|)$$

with $C = C(\|f\|, \|u_d\|, \|q_d\|, \alpha)$. Hence, with the Lemmas 4.3, 4.5, 5.16 and 5.17 as well as (5.3) we get

$$\begin{aligned}\|\bar q - \tilde q_h\| &\le C(\|S_h(\bar q_h) - S_h(R_h\bar q)\| + \|S_h(R_h\bar q) - S_h(\bar q)\| + \|S_h(\bar q) - S(\bar q)\| \\ &\quad + \|Z_h(\bar q_h) - Z_h(R_h\bar q)\| + \|Z_h(R_h\bar q) - Z_h(\bar q)\| + \|Z_h(\bar q) - Z(\bar q)\| + \|q_d - I_h q_d\|) \\ &\le C(\|\bar q_h - R_h\bar q\| + h^2) \\ &\le Ch^2\end{aligned}$$

with

$$C = C(\tilde C, \|f\|, \|u_d\|, \|q_d\|, \|\nabla q_d\|, \|\nabla^2 q_d\|, \|\nabla S(\bar q)\|_\infty, \|\nabla Z(\bar q)\|_\infty, \|\nabla q_d\|_\infty, \alpha). \quad \square$$

Remark 5.19. Alternatively to the post-processing technique discussed above, the variational approach originally introduced by Hinze, see [14], is transferable to the optimal control problem under consideration to obtain quadratic order of convergence in the control with respect to the L^2-norm. The basic idea is not to discretize the control variable. The solution $\hat q_h$ is then not a mesh-function and there holds

$$\hat q_h = P_{[a,b]}\left(\frac{1}{\alpha}S_h(\hat q_h)Z_h(\hat q_h) + I_h q_d\right). \tag{5.25}$$

It is a short proof to verify, that $\|\bar q - \hat q_h\| = \mathcal{O}(h^2)$. However, this ansatz requires a non-standard implementation which goes beyond the implementation for linear–quadratic problems. This is due to the fact that the term $\frac{1}{\alpha}S_h(\hat q_h)Z_h(\hat q_h)$ in (5.25) is cellwise bi-quadratic and that as a consequence of the projection the boundaries of the active sets are in general curved lines.

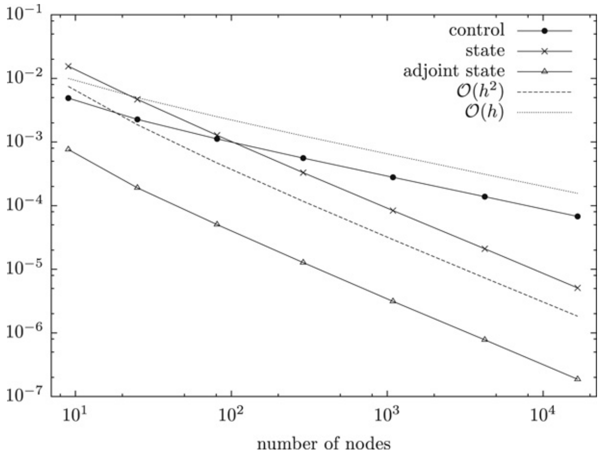

Fig. 1. Example 6.1: Discretization error of the control, state and adjoint state when discretizing the control by cellwise constants.

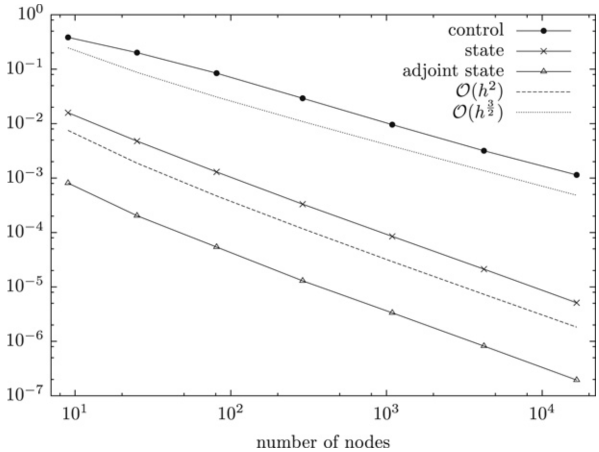

Fig. 2. Example 6.1: Discretization error of the control, state and adjoint state when discretizing the control by continuous cellwise bilinear finite elements.

6. Numerical examples

In this section we are going to confirm the a priori error estimates for the error in the control numerically. Thereby the optimal control problem is solved by the optimization library RoDoBo [27] and the finite element toolkit Gascoigne [28] using a primal–dual active set strategy (cf. [29–31]) in combination with a conjugate gradient method applied to the reduced problem (3.3). In the first example we consider an optimal control problem with an unknown exact solution, in the second one the analytical solution is known. In both cases let $\Omega = (0, 1)^2$ and $x = (x_1, x_2) \in \Omega$.

Example 6.1. We consider the following concretization of the optimal control problem (P):

$$u_d(x) = 5 \times 10^{-4} \cdot \mathrm{e}^{-x_1}, \qquad q_d(x) = 0, \qquad f(x) = x_1^{-\frac{1}{4}}, \qquad \alpha = 10^{-4}, \qquad a = 1, \qquad b = 2.$$

To calculate the error in the control for Example 6.1 we compare the solutions with the solution which we calculated on an eight times uniformly refined mesh. The Figs. 1 and 2 depict the development of the L^2-error in the control under uniform refinement of the mesh. In Fig. 1, the expected order $\mathcal{O}(h)$ for cellwise constant control is observed and in Fig. 2, the order $\mathcal{O}(h^{\frac{3}{2}})$ for bilinear control discretization is shown. From the numerical solution we can derive, that in both cases the control constraints are active. Additionally, the Figs. 1 and 2 show the L^2-error in the state and adjoint state. Thereby, we observe convergence of order $\mathcal{O}(h^2)$ regardless of the type of discretization used for the control. Since the post-processing strategy presented in Section 5.3 relies essentially on the convergence properties of the state and adjoint state variable, Fig. 1 confirms the order of convergence for the post-processing strategy proved in Section 5.3.

In the next example we consider a concretization of (P) whose analytical solution is known.

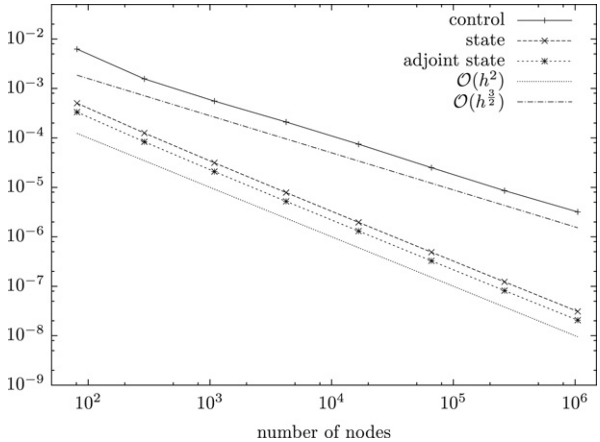

Fig. 3. Example 6.2: Discretization error of the control, state and adjoint state when discretizing the control by continuous cellwise bilinear finite elements.

Example 6.2. Let

$$
\begin{aligned}
u_d(x) &= (1-x_1)(1-x_2) - 0.1 \cdot \pi^2 \cdot \sin(\pi x_1)\sin(\pi x_2) \\
&\quad - 0.05 \cdot \sin(\pi x_1)\sin(\pi x_2) \cdot P_{[a,b]}\left(5 \cdot \sin(\pi x_1)\sin(\pi x_2)(1-x_1)(1-x_2)\right), \\
f(x) &= 2(x_2(1-x_2) + x_1(1-x_1)) \\
&\quad + (1-x_1)(1-x_2) \cdot P_{[a,b]}\left(5 \cdot \sin(\pi x_1)\sin(\pi x_2)(1-x_1)(1-x_2)\right), \\
q_d(x) &= 0, \qquad \alpha = 0.01, \qquad a = 0.1, \qquad b = 0.3.
\end{aligned}
$$

Then we have for (P) the following optimal state, adjoint state and control:

$$
\begin{aligned}
&\bar{u}(x) = (1-x_1)(1-x_2), \\
&\bar{z}(x) = 0.05 \cdot \sin(\pi x_1)\sin(\pi x_2), \\
&\bar{q} = P_{[a,b]}\left(\frac{1}{\alpha}\bar{z}\bar{u}\right).
\end{aligned}
$$

In Fig. 3 the L^2-error in the control, state and adjoint state under uniform refinement of the mesh for the data given in Example 6.2 is shown, when discretizing the control with continuous cellwise bilinear finite elements. Here, we calculate the error by comparison with the analytical solution. Again, we derive from the numerical solution, that the control constraints are active and we see the order $\mathcal{O}(h^{\frac{3}{2}})$ for the control and $\mathcal{O}(h^2)$ for the state and adjoint state.

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Appendix

Proof of Lemma 5.15. By means of the L^2-projection $\pi_h: Q \to Q_h$, we split

$$(v_h w_h, \bar{q} - R_h\bar{q}) = (v_h w_h, \bar{q} - \pi_h\bar{q}) + (v_h w_h, \pi_h\bar{q} - R_h\bar{q}). \tag{A.1}$$

For the first term we obtain

$$
\begin{aligned}
(v_h w_h, \bar{q} - \pi_h\bar{q}) &= (v_h w_h - \pi_h(v_h w_h), \bar{q} - \pi_h\bar{q}) \\
&\leq Ch^2 \|\nabla(v_h w_h)\| \|\nabla\bar{q}\| \\
&\leq Ch^2 (\|\nabla v_h w_h\| + \|v_h \nabla w_h\|)\|\nabla\bar{q}\|
\end{aligned}
$$

and hence, for all $r \geq 3$

$$(v_h w_h, \bar{q} - \pi_h\bar{q}) \leq \frac{C}{\alpha} h^2 \|\nabla v_h\| \|w_h\|_\infty \|\nabla(S(\bar{q})Z(\bar{q})) + \nabla q_d\| + \frac{C}{\alpha} h^2 \|\nabla v_h\| \|w_h\|_{1,r} \|\nabla(S(\bar{q})Z(\bar{q})) + \nabla q_d\|.$$

Utilizing the fact that $\pi_h\bar{q}$ as well as $R_h\bar{q}$ are constant on each cell K, we obtain for the second term in (A.1)

$$\begin{aligned}(v_h w_h, \pi_h\bar{q} - R_h\bar{q}) &= \sum_K \int_K v_h w_h(\pi\bar{q} - R_h\bar{q})\mathrm{d}x \\ &= \sum_K \frac{1}{|K|}\int_K v_h w_h \mathrm{d}x \int_K (\pi\bar{q} - R_h\bar{q})\mathrm{d}x \\ &\le \|w_h\|_\infty \|v_h\|_\infty \sum_K \left|\int_K (\bar{q} - R_h\bar{q})\mathrm{d}x\right|.\end{aligned}$$

As in Section 5.2, we split the last sum using the separation $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^2 \cup \mathcal{T}_h^3$. For the sum $\mathcal{T}_h^1 \cup \mathcal{T}_h^2$ we obtain by means of (5.16) and the fact that $\bar{q}$ equals either a, b or $\frac{1}{\alpha}S(\bar{q})Z(\bar{q}) + q_d$:

$$\begin{aligned}\sum_{K\in\mathcal{T}_h^1\cup\mathcal{T}_h^2}\left|\int_K (\bar{q} - R_h\bar{q})\mathrm{d}x\right| &\le Ch^2 \sum_{K\in\mathcal{T}_h^1\cup\mathcal{T}_h^2} |K|^{\frac{1}{2}} \|\nabla^2\bar{q}\|_{L^2(K)} \\ &\le Ch^2 \left(\sum_{K\in\mathcal{T}_h^1\cup\mathcal{T}_h^2}|K|\right)^{\frac{1}{2}} \|\nabla^2\bar{q}\|_{L^2(\mathcal{T}_h^1\cup\mathcal{T}_h^2)} \\ &\le Ch^2\|\nabla^2(S(\bar{q})Z(\bar{q}) + q_d)\|_{L^2(\Omega)}.\end{aligned}$$

For the part of the sum over $\mathcal{T}_h^3$, estimate (5.17) and Assumption 5.7 leads to

$$\begin{aligned}\sum_{K\in\mathcal{T}_h^3}\left|\int_K (\bar{q} - R_h\bar{q})dx\right| &\le \|\bar{q} - R_h\bar{q}\|_{L^\infty(\mathcal{T}_h^3)} \sum_{K\in\mathcal{T}_h^3}|K| \\ &\le Ch\|\nabla\bar{q}\|_{L^\infty(\mathcal{T}_h^3)}\sum_{K\in\mathcal{T}_h^3}|K| \\ &\le \frac{C}{\alpha}h^2\|\nabla(S(\bar{q})Z(\bar{q}) + q_d)\|_{L^\infty(\Omega)}.\end{aligned}$$

We obtain

$$(v_h w_h, \pi_h\bar{q} - R_h\bar{q}) \le Ch^2\|w_h\|_\infty\|v_h\|_\infty$$

with

$$C = C(\|f\|, \|u_d\|, \|q_d\|, \|\nabla q_d\|, \|\nabla^2 q_d\|, \|\nabla S(\bar{q})\|_\infty, \|\nabla Z(\bar{q})\|_\infty, \|\nabla q_d\|_\infty, \alpha).$$

This completes the proof. $\square$

Proof of Lemma 5.17. The existence of a $h_0 > 0$ and a sequence $(\bar{q}_h)_{0<h<h_0}$ of discrete local solutions with $\|\bar{q} - \bar{q}_h\| = \mathcal{O}(h)$ follows immediately from Section 5.1.

Now, we prove the error estimate (5.23). For every $\epsilon > 0$ there exists a $h_0 > 0$, such that

$$\|R_h\bar{q} - \bar{q}\| \le \varepsilon \quad \text{and} \quad \|\bar{q}_h - \bar{q}\| \le \varepsilon$$

for $h \le h_0$. Hence, we can apply (4.7) and get with $\xi = tR_h\bar{q} + (1-t)\bar{q}_h$ for a $t \in [0, 1]$

$$\frac{\gamma}{4}\|R_h\bar{q} - \bar{q}_h\|^2 \le j_h''(\xi)(R_h\bar{q} - \bar{q}_h, R_h\bar{q} - \bar{q}_h) = j_h'(R_h\bar{q})(R_h\bar{q} - \bar{q}_h) - j_h'(\bar{q}_h)(R_h\bar{q} - \bar{q}_h)$$

for h sufficiently small. From the optimality condition and Lemma 5.13 we have

$$-j_h'(\bar{q}_h)(R_h\bar{q} - \bar{q}_h) \le 0 \le -(\alpha R_h\bar{q} + R_h(Z(\bar{q})S(\bar{q})) + R_h q_d, R_h\bar{q} - \bar{q}_h)$$

and hence,

$$\begin{aligned}\frac{\gamma}{4}\|R_h\bar{q} - \bar{q}_h\|^2 &\le j_h'(R_h\bar{q})(R_h\bar{q} - \bar{q}_h) - (\alpha R_h\bar{q} + R_h(Z(\bar{q})S(\bar{q})), R_h\bar{q} - \bar{q}_h) - (R_h q_d, R_h\bar{q} - \bar{q}_h) \\ &\le (Z_h(R_h\bar{q})S_h(R_h\bar{q}) - Z(\bar{q})S(\bar{q}), R_h\bar{q} - \bar{q}_h) + (Z(\bar{q})S(\bar{q}) \\ &\quad - R_h(Z(\bar{q})S(\bar{q})), R_h\bar{q} - \bar{q}_h) + (q_d - R_h q_d, R_h\bar{q} - \bar{q}_h) \\ &=: A_1 + A_2 + A_3.\end{aligned}$$

Now, we further estimate the terms A_1, A_2 and A_3:

$$\begin{aligned} A_1 &= \int_\Omega (Z_h(R_h\bar{q}) - Z(\bar{q}))S_h(R_h\bar{q})(R_h\bar{q} - \bar{q}_h)\mathrm{d}x + \int_\Omega Z(\bar{q})(S_h(R_h\bar{q}) - S(\bar{q}))(R_h\bar{q} - \bar{q}_h)\mathrm{d}x \\ &=: B_1 + B_2. \end{aligned}$$

We start with the consideration of B_2:

$$\begin{aligned} B_2 &\leq \|Z(\bar{q})\|_\infty \|S_h(R_h\bar{q}) - S(\bar{q})\| \|R_h\bar{q} - \bar{q}_h\| \\ &\leq \|Z(\bar{q})\|_\infty (\|S_h(R_h\bar{q}) - S_h(\bar{q})\| + \|S_h(\bar{q}) - S(\bar{q})\|) \|R_h\bar{q} - \bar{q}_h\|. \end{aligned}$$

Applying the Lemmas 5.16 and 4.3 we deduce

$$B_2 \leq Ch^2 \|R_h\bar{q} - \bar{q}_h\|$$

with

$$C = C(\tilde{C}, \|f\|, \|u_d\|, \|q_d\|, \|\nabla q_d\|, \|\nabla^2 q_d\|, \|\nabla S(\bar{q})\|_\infty, \|\nabla Z(\bar{q})\|_\infty, \|\nabla q_d\|_\infty, \alpha).$$

In the same way, we get for B_1:

$$\begin{aligned} B_1 &\leq \|S_h(R_h\bar{q})\|_\infty (\|Z_h(R_h\bar{q}) - Z_h(\bar{q})\| + \|Z_h(\bar{q}) - Z(\bar{q})\|) \|R_h\bar{q} - \bar{q}_h\| \\ &\leq Ch^2 \|R_h\bar{q} - \bar{q}_h\|. \end{aligned}$$

To estimate A_2 we apply Lemma 5.14 with $p_h = R_h\bar{q} - \bar{q}_h$ and $v = Z(\bar{q})S(\bar{q})$ we have:

$$(Z(\bar{q})S(\bar{q}) - R_h(Z(\bar{q})S(\bar{q})), R_h\bar{q} - \bar{q}_h) \leq Ch^2 \|R_h\bar{q} - \bar{q}_h\| \|\nabla^2(S(\bar{q})Z(\bar{q}))\|.$$

Finally, we have by Lemma 5.14

$$A_3 \leq Ch^2 \|\nabla^2 q_d\| \|R_h\bar{q} - \bar{q}_h\|.$$

By combining these estimates, we obtain the asserted estimate. □

References

[1] Harvey T. Banks, Karl Kunisch, Estimation Techniques for Distributed Parameter Systems, Birkhäuser, Boston, 1989.
[2] Richard S. Falk, Error estimates for the numerical identification of a variable coefficient, Math. Comp. 40 (1983) 537–546.
[3] Tommi Kärkkäinen, Error estimates for distributed parameter identification in linear elliptic equations, J. Math. Syst. Estim. Control 6 (1996) 1–20.
[4] Tommi Kärkkäinen, A linearization technique and error estimates for distributed parameter identification in quasilinear problems, Numer. Funct. Anal. Optim. 17 (1996) 345–364.
[5] Pekka Neittaanmäki, Xue-Cheng Tai, A linear approach for the nonlinear distributed parameter identification problem, Internat. Ser. Numer. Math. 100 (1991).
[6] Pekka Neittaanmäki, Xue-Cheng Tai, On the numerical solution of the distributed parameter identification problem, in: Estimation and Control of Distributed Parameter Systems, in: International Series of Numerical Mathematics, Birkhäuser, Basel, 1991, pp. 317–330.
[7] Pekka Neittaanmäki, Xue-Cheng Tai, Error estimates for numerical identification of distributed parameters, J. Comput. Appl. Math. 10 (1992) 66–78.
[8] Pekka Neittaanmäki, Xue-Cheng Tai, Pointwise error estimates for distributed parameter identification, in: D. Bainov (Ed.), Proceedings of the 7th International Colloquium on Differential Equations, Plovdiv, Bulgaria, August 18–23, 1996, International Science Publishers, Utrecht, 1997, pp. 455–468.
[9] Nadir Arada, Eduardo Casas, Fredi Tröltzsch, Error estimates for the numerical approximation of a semilinear elliptic control problem, Comput. Optim. Appl. 23 (2002) 201–229.
[10] Roland Becker, Boris Vexler, Optimal control of the convection-diffusion equation using stabilized finite element methods, Numer. Math. 106 (2007) 349–367.
[11] Eduardo Casas, Using piecewise linear functions in the numerical approximation of semilinear elliptic control problems, Adv. Comput. Math. 26 (2007) 137–153.
[12] Eduardo Casas, Fredi Tröltzsch, Error estimates for the finite-element approximation of a semilinear elliptic control problem, Control Cybernet. 31 (2002) 695–712.
[13] Eduardo Casas, Fredi Tröltzsch, Error estimates for linear–quadratic elliptic control problems, in: Viorel Barbu, et al. (Eds.), Analysis and Optimization of Differential Systems. IFIP TC7/WG 7.2 International Working Conference, Constanta, Romania, Kluwer Academic Publishers, Boston, MA, 2003.
[14] Michael Hinze, A variational discretization concept in control constrained optimization: The linear–quadratic case, Comput. Optim. Appl. 30 (2005) 45–61.
[15] Christian Meyer, Arnd Rösch, Superconvergence properties of optimal control problems, SIAM J. Control Optim. 43 (2004) 970–985.
[16] Kazimierz Malanowski, Convergence of approximations vs. regularity of solutions for convex, control-constrained optimal-control problems, Appl. Math. Optim. 8 (1981) 69–95.
[17] Dominik Meidner, Boris Vexler, A priori error estimates for space-time finite element discretization of parabolic optimal control problems. Part I: Problems without control constraints, SIAM J. Control Optim. 47 (2008) 1150–1177.
[18] Dominik Meidner, Boris Vexler, A priori error estimates for space-time finite element discretization of parabolic optimal control problems. Part II: Problems with control constraints, SIAM J. Control Optim. 47 (2008) 1301–1329.
[19] Jacques Louis Lions, Optimal Control of Systems Governed by Partial Differential Equations, in: Grundlehren Math. Wiss., vol. 170, Springer, Berlin, 1971.
[20] Fredi Tröltzsch, Optimale Steuerung Partieller Differentialgleichungen, Vieweg, Wiesbaden, 2005.
[21] Pierre Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman, Boston, 1985.
[22] Vladimir A. Kozlov, Vladimir G. Maz'ja, Jürgen Roßmann, Spectral Problems Associated with Corner Singularities of Solutions of Elliptic Equations, American Mathematical Society, Providence, RI, 2001.
[23] Arnd Rösch, Daniel Wachsmuth, Numerical verification of optimality conditions, SIAM J. Control Optim. 47 (2008) 2557–2581.

[24] Alexander Ern, Jean-Luc Guermond, Theory and Practice of Finite Elements, Springer, New York, 2004.
[25] Dietrich Braess, Finite Elemente Theorie, Springer, Berlin, 2003.
[26] Susanne C. Brenner, L. Ridgway scott, The Mathematical Theory of Finite Element Methods, Springer, New York, 2002.
[27] RoDoBo. A C++ library for optimization with stationary and nonstationary PDEs. http://www.rodobo.uni-hd.de.
[28] GASCOIGNE, The Finite Element Toolkit. http://www.gascoigne.uni-hd.de.
[29] Maïtine Bergounioux, Kazufumi Ito, Karl Kunisch, Primal–dual strategy for constrained optimal control problems, SIAM J. Control Optim. 37 (1999) 1176–1194.
[30] Kazufumi Ito, Karl Kunisch, The primal–dual active set method for nonlinear optimal control problems with bilateral constraints, SIAM J. Control Optim. 43 (2004) 357–376.
[31] Karl Kunisch, Arnd Rösch, Primal–dual active set strategy for a general class of constrained optimal control problems, SIAM J. Optim. 13 (2002) 321–334.